3

Elasticity and Flexure

3.1 Introduction

In the previous chapter we introduced the concepts of stress and strain. For many solids it is appropriate to relate stress to strain through the laws of elasticity. *Elastic materials* deform when a force is applied and return to their original shape when the force is removed. Almost all solid materials, including essentially all rocks at relatively low temperatures and pressures, behave elastically when the applied forces are not too large. In addition, the elastic strain of many rocks is linearly proportional to the applied stress. The equations of linear elasticity are greatly simplified if the material is *isotropic*, that is, if its elastic properties are independent of direction. Although some metamorphic rocks with strong foliations are not strictly isotropic, the isotropic approximation is usually satisfactory for the earth's crust and mantle.

At high stress levels, or at temperatures that are a significant fraction of the rock solidus, deviations from elastic behavior occur. At low temperatures and confining pressures, rocks are brittle solids, and large deviatoric stresses cause fracture. As rocks are buried more deeply in the earth, they are subjected to increasingly large confining pressures due to the increasing weight of the overburden. When the confining pressure on the rock approaches its brittle failure strength, it deforms plastically. *Plastic deformation* is a continuous, irreversible deformation without fracture. If the applied force causing plastic deformation is removed, some fraction of the deformation remains. We consider plastic deformation in Section 7–11. As discussed in Chapter 1, hot mantle rocks behave as a fluid on geological time scales; that is, they continuously deform under an applied force.

Given that rocks behave quite differently in response to applied forces, depending on conditions of temperature and pressure, it is important to determine what fraction of the rocks of the crust and upper mantle behave

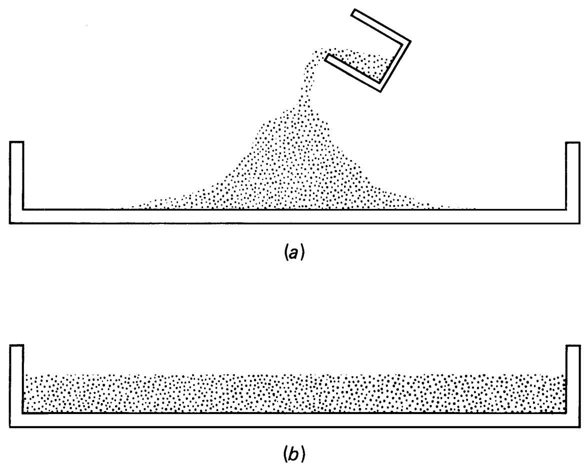

Figure 3.1 (*a*) Structure formed immediately after rapidly pouring a very viscous fluid into a container. (*b*) Final shape of the fluid after a long time has elapsed.

elastically on geological time scales. One of the fundamental postulates of plate tectonics is that the surface plates constituting the lithosphere do not deform significantly on geological time scales. Several observations directly confirm this postulate. We know that the transform faults connecting offset segments of the oceanic ridge system are responsible for the major linear fracture zones in the ocean. That these fracture zones remain linear and at constant separation is direct evidence that the oceanic lithosphere does not deform on a time scale of 10^8 years. Similar evidence comes from the linearity of the magnetic lineaments of the seafloor (see Section 1–8).

There is yet other direct evidence of the elastic behavior of the lithosphere on geological time scales. Although erosion destroys mountain ranges on a time scale of 10^6 to 10^7 years, many geological structures in the continental crust have ages greater than 10^9 years. The very existence of these structures is evidence of the elastic behavior of the lithosphere. If the rocks of the crust behaved as a fluid on geological time scales, the gravitational body force would have erased these structures. As an example, pour a very viscous substance such as molasses onto the bottom of a flat pan. If the fluid is sufficiently viscous and is poured quickly enough, a structure resembling a mountain forms (see Figure 3–1*a*). However, over time, the fluid will eventually cover the bottom of the pan to a uniform depth (see Figure 3–1*b*). The gravitational body force causes the fluid to flow so as to minimize the gravitational potential energy.

A number of geological phenomena allow the long-term elastic behavior of the lithosphere to be studied quantitatively. In several instances the lithosphere bends under surface loads. Direct evidence of this bending comes from the Hawaiian Islands and many other island chains, individual islands, and seamounts. There is also observational evidence of the elastic bending of the oceanic lithosphere at ocean trenches and of the continental lithosphere at sedimentary basins – the Michigan basin, for example. We will make quantitative comparisons of the theoretically predicted elastic deformations of these structures with the observational data in later sections of this chapter.

One important reason for studying the elastic behavior of the lithosphere is to determine the state of stress in the lithosphere. This stress distribution is responsible for the occurrence of earthquakes. Earthquakes are direct evidence of high stress levels in the lithosphere. An earthquake relieves accumulated strain in the lithosphere. The presence of mountains is also evidence of high stress levels. Elastic stresses must balance the gravitational body forces on mountains. Because of their elastic behavior, surface plates can transmit stresses over large horizontal distances.

3.2 Linear Elasticity

A linear, isotropic, elastic solid is one in which stresses are linearly proportional to strains and mechanical properties have no preferred orientations. The principal axes of stress and strain coincide in such a medium, and the connection between stress and strain can be conveniently written in this coordinate system as

$$\sigma_1 = (\lambda + 2G)\varepsilon_1 + \lambda\varepsilon_2 + \lambda\varepsilon_3 \tag{3.1}$$

$$\sigma_2 = \lambda\varepsilon_1 + (\lambda + 2G)\varepsilon_2 + \lambda\varepsilon_3 \tag{3.2}$$

$$\sigma_3 = \lambda\varepsilon_1 + \lambda\varepsilon_2 + (\lambda + 2G)\varepsilon_3, \tag{3.3}$$

where the material properties λ and G are known as *Lamé parameters*; G is also known as the *modulus of rigidity*. The material properties are such that a principal strain component ε produces a stress $(\lambda+2G)\varepsilon$ in the same direction and stresses $\lambda\varepsilon$ in mutually perpendicular directions.

Equations (3–1) to (3–3) can be written in the inverse form as

$$\varepsilon_1 = \frac{1}{E}\sigma_1 - \frac{\nu}{E}\sigma_2 - \frac{\nu}{E}\sigma_3 \tag{3.4}$$

$$\varepsilon_2 = -\frac{\nu}{E}\sigma_1 + \frac{1}{E}\sigma_2 - \frac{\nu}{E}\sigma_3 \tag{3.5}$$

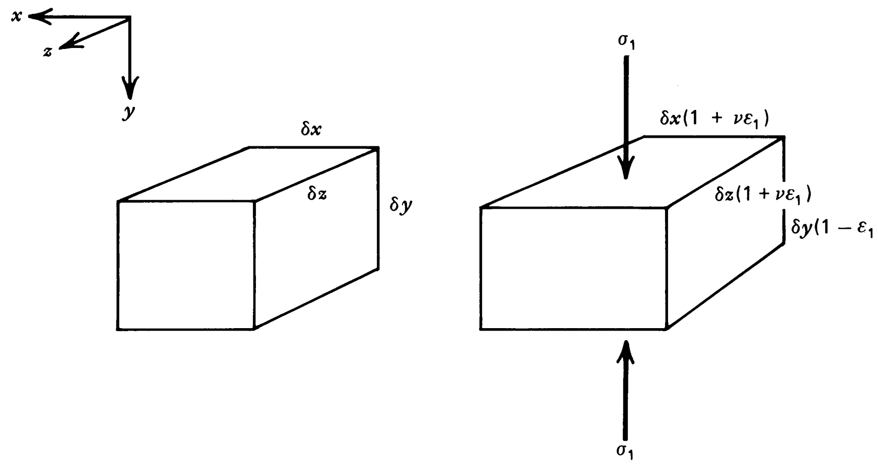

Figure 3.2 Deformation under uniaxial stress.

$$\varepsilon_3 = -\frac{\nu}{E}\sigma_1 - \frac{\nu}{E}\sigma_2 + \frac{1}{E}\sigma_3, \tag{3.6}$$

and E and ν are material properties known as *Young's modulus* and *Poisson's ratio*, respectively. A principal stress component σ produces a strain σ/E in the same direction and strains $(-\nu\sigma/E)$ in mutually orthogonal directions.

The elastic behavior of a material can be characterized by specifying either λ and G or E and ν; the sets of parameters are not independent. Analytic formulas expressing λ and G in terms of E and ν, and vice versa, are obtained in the following sections. Values of E, G, and ν for various rocks are given in Section E of Appendix 2. Young's modulus of rocks varies from about 10 to 100 GPa, and Poisson's ratio varies between 0.1 and 0.4. The elastic properties of the earth's mantle and core can be obtained from seismic velocities and the density distribution. The elastic properties E, G, and ν inferred from a typical seismically derived earth model are given in Section F of Appendix 2. The absence of shear waves in the outer core ($G = 0$) is taken as conclusive evidence that the outer core is a liquid. In the outer core ν has the value 0.5, which we will see is appropriate to an incompressible fluid.

The behavior of linear solids is more readily illustrated if we consider idealized situations where several of the stress and strain components vanish. These can then be applied to important geological problems.

3.3 Uniaxial Stress

In a state of *uniaxial stress* only one of the principal stresses, σ_1 say, is nonzero. Under this circumstance Equations (3–2) and (3–3), with $\sigma_2 = \sigma_3 = 0$, give

$$\varepsilon_2 = \varepsilon_3 = \frac{-\lambda}{2(\lambda + G)}\varepsilon_1. \tag{3.7}$$

Not only does the stress σ_1 produce a strain ε_1, but it changes the linear dimensions of elements aligned perpendicular to the axis of stress. If σ_1 is a compression, then ε_1 is a decrease in length, and both ε_2 and ε_3 are increases in length. The element in Figure 3–2 has been shortened in the y direction, but its cross section in the xz plane has expanded.

Using Equations (3–4) to (3–6), we can also write

$$\varepsilon_2 = \varepsilon_3 = -\frac{\nu}{E}\sigma_1 = -\nu\varepsilon_1. \tag{3.8}$$

By comparing Equations (3–7) and (3–8), we see that

$$\nu = \frac{\lambda}{2(\lambda + G)}. \tag{3.9}$$

From Equations (3–1) and (3–7) we find

$$\sigma_1 = \frac{G(3\lambda + 2G)}{(\lambda + G)}\varepsilon_1, \tag{3.10}$$

which, with the help of Equation (3–8), identifies Young's modulus as

$$E = \frac{G(3\lambda + 2G)}{(\lambda + G)}. \tag{3.11}$$

Equations (3–9) and (3–11) can be inverted to yield the following formulas for G and λ in terms of E and ν

$$G = \frac{E}{2(1 + \nu)} \tag{3.12}$$

$$\lambda = \frac{E\nu}{(1 + \nu)(1 - 2\nu)}. \tag{3.13}$$

The relation between stress and strain in uniaxial compression or tension from Equation (3–8),

$$\sigma_1 = E\varepsilon_1, \tag{3.14}$$

is also known as *Hooke's law.* A linear elastic solid is said to exhibit Hookean behavior. Uniaxial compression testing in the laboratory is one of the simplest methods of determining the elastic properties of rocks. Figure 3–3

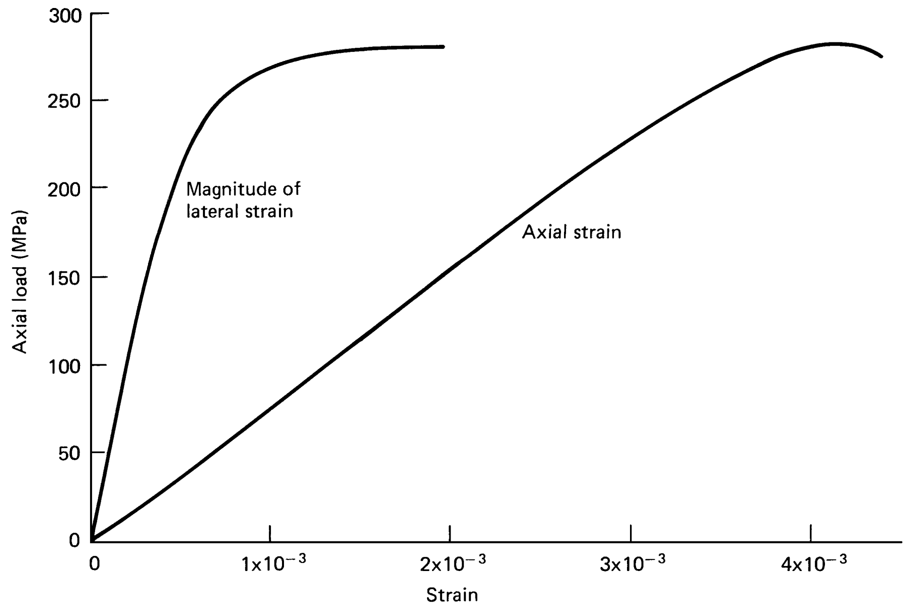

Figure 3.3 Stress–strain curves for quartzite in uniaxial compression (Bieniawski, 1967).

shows the data from such a test on a cylindrical sample of quartzite. The rock deforms approximately elastically until the applied stress exceeds the compressive strength of the rock, at which point failure occurs. Compressive strengths of rocks are hundreds to thousands of megapascals. As we discussed in the previous chapter, a typical tectonic stress is 10 MPa. With $E = 70$ GPa, this yields a typical tectonic strain in uniaxial stress of 1.4×10^{-4}.

The *dilatation* Δ or fractional volume change in uniaxial compression is, according to Equation (3–8),

$$\Delta = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon_1(1 - 2\nu). \tag{3.15}$$

The decrease in volume due to contraction in the direction of compressive stress is offset by an increase in volume due to expansion in the orthogonal directions. Equation (3–15) allows us to determine Poisson's ratio for an *incompressible material*, which cannot undergo a net change in volume. In order for Δ to equal zero in uniaxial compression, ν must equal 1/2. Under uniaxial compression, an incompressible material contracts in the direction of applied stress but expands exactly half as much in each of the perpendicular directions.

There are some circumstances in which the formulas of uniaxial compression can be applied to calculate the strains in rocks. Consider, for example, a rectangular column of height h that is free to expand or contract in the horizontal; that is, it is laterally unconstrained. By this we mean that the horizontal stresses are zero ($\sigma_2 = \sigma_3 = 0$). Then the vertical stress σ_1 at a distance y from the top of the column of rock is given by the weight of the column,

$$\sigma_1 = \rho g y. \tag{3.16}$$

The vertical strain as a function of the distance y from the top is

$$\varepsilon_1 = \frac{\rho g y}{E}. \tag{3.17}$$

The slab contracts in the vertical by an amount

$$\delta h = \int_0^h \varepsilon_1 \, dy = \frac{\rho g}{E} \int_0^h y \, dy = \frac{\rho g h^2}{2E}. \tag{3.18}$$

3.4 Uniaxial Strain

The state of *uniaxial strain* corresponds to only one nonzero component of principal strain, ε_1 say. With $\varepsilon_2 = \varepsilon_3 = 0$, Equations (3–1) to (3–3) give

$$\sigma_1 = (\lambda + 2G)\varepsilon_1 \tag{3.19}$$

$$\sigma_2 = \sigma_3 = \lambda\varepsilon_1 = \frac{\lambda}{(\lambda + 2G)}\sigma_1. \tag{3.20}$$

Equations (3–4) to (3–6) simplify to

$$\sigma_2 = \sigma_3 = \frac{\nu}{(1 - \nu)}\sigma_1 \tag{3.21}$$

$$\sigma_1 = \frac{(1 - \nu)E\varepsilon_1}{(1 + \nu)(1 - 2\nu)}. \tag{3.22}$$

By comparing Equations (3–19) to (3–22), one can also derive the relations already given between λ, G and ν, E.

The equations of uniaxial strain can be used to determine the change in stress due to sedimentation or erosion. We first consider sedimentation and assume that an initial surface is covered by h km of sediments of density ρ, as shown in Figure 3–4. We also assume that the base of the new sedimentary basin is laterally confined so that the equations of uniaxial strain are applicable. The two horizontal components of strain are zero, $\varepsilon_2 = \varepsilon_3 = 0$. The vertical principal stress on the initial surface σ_1 is given by the weight of the overburden

$$\sigma_1 = \rho g h. \tag{3.23}$$

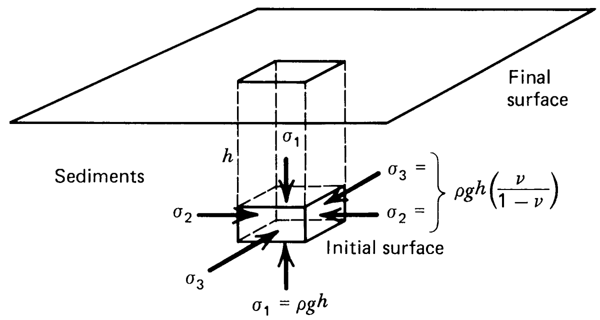

Figure 3.4 Stresses on a surface covered by sediments of thickness h.

From Equation (3–21) the horizontal normal stresses are given by

$$\sigma_2 = \sigma_3 = \frac{\nu}{(1-\nu)}\rho g h. \tag{3.24}$$

The horizontal stresses are also compressive, but they are smaller than the vertical stress.

It is of interest to determine the deviatoric stresses after sedimentation. The pressure at depth h as defined by Equation (2–61) is

$$p = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{(1+\nu)}{3(1-\nu)}\rho g h. \tag{3.25}$$

The deviatoric stresses are then determined from Equations (2–63) with the result

$$\sigma_1' = \sigma_1 - p = \frac{2(1-2\nu)}{3(1-\nu)}\rho g h \tag{3.26}$$

$$\sigma_2' = \sigma_2 - p = \sigma_3' = \sigma_3 - p = -\frac{(1-2\nu)}{3(1-\nu)}\rho g h. \tag{3.27}$$

The horizontal deviatoric stress is tensional. For $\nu = 0.25$ the horizontal deviatoric stress is 2/9 of the lithostatic stress. With $\rho = 3000$ kg m^{-3} and $h = 2$ km the horizontal deviatoric stress is -13.3 MPa. This stress is of the same order as measured surface stresses.

We next consider erosion. If the initial state of stress before erosion is that given above, erosion will result in the state of stress that existed before sedimentation occurred. The processes of sedimentation and erosion are reversible. However, in many cases the initial state of stress prior to erosion is lithostatic. Therefore at a depth h the principal stresses are

$$\sigma_1 = \sigma_2 = \sigma_3 = \rho g h. \tag{3.28}$$

After the erosion of h km of overburden the vertical stress at the surface is $\bar{\sigma}_1 = 0$ (an overbar denotes a stress after erosion). The change in vertical stress $\Delta\sigma_1 = \bar{\sigma}_1 - \sigma_1$ is $-\rho g h$. If only ε_1 is nonzero, Equation (3–21) gives

$$\Delta\sigma_2 = \Delta\sigma_3 = \left(\frac{\nu}{1-\nu}\right)\Delta\sigma_1. \tag{3.29}$$

The horizontal surface stresses after erosion $\bar{\sigma}_2$ and $\bar{\sigma}_3$ are consequently given by

$$\begin{aligned}\bar{\sigma}_2 = \bar{\sigma}_3 = \sigma_2 + \Delta\sigma_2 &= \rho g h - \frac{\nu}{(1-\nu)}\rho g h \\ &= \left(\frac{1-2\nu}{1-\nu}\right)\rho g h. \end{aligned} \tag{3.30}$$

If $h = 5$ km, $\nu = 0.25$, and $\rho = 3000$ kg m^{-3}, we find from Equation (3–30) that $\bar{\sigma}_2 = \bar{\sigma}_3 = 100$ MPa. Erosion can result in large surface compressive stresses due simply to the elastic behavior of the rock. This mechanism is one explanation for the widespread occurrence of near-surface compressive stresses in the continents.

Problem 3.1 Determine the surface stress after the erosion of 10 km of granite. Assume that the initial state of stress is lithostatic and that $\rho = 2700$ kg m^{-3} and $\nu = 0.25$.

Problem 3.2 An unstressed surface is covered with sediments with a density of 2500 kg m^{-3} to a depth of 5 km. If the surface is laterally constrained and has a Poisson's ratio of 0.25, what are the three components of stress at the original surface?

Problem 3.3 A horizontal stress σ_1 may be accompanied by stress in other directions. If it is assumed that there is no displacement in the other horizontal direction and zero stress in the vertical, find the stress σ_2 in the other horizontal direction and the strain ε_3 in the vertical direction.

Problem 3.4 Assume that the earth is unconstrained in one lateral direction ($\sigma_2 = \sigma_3$) and is constrained in the other ($\varepsilon_1 = 0$). Determine ε_2 and σ_1 when y kilometers of rock of density ρ are eroded away. Assume that the initial state of stress was lithostatic.

3.5 Plane Stress

The state of *plane stress* exists when there is only one zero component of principal stress; that is, $\sigma_3 = 0$, $\sigma_1 \neq 0$, $\sigma_2 \neq 0$. The situation is sketched

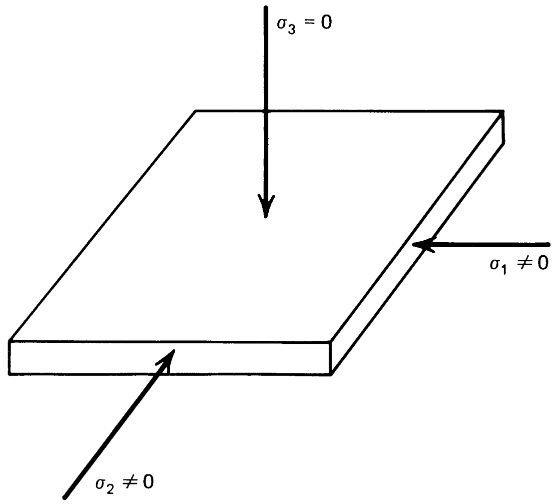

Figure 3.5 Plane stress.

in Figure 3–5, which shows a thin plate loaded on its edges. The strain components according to Equations (3–4) to (3–6) are

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2) \tag{3.31}$$

$$\varepsilon_2 = \frac{1}{E}(\sigma_2 - \nu\sigma_1) \tag{3.32}$$

$$\varepsilon_3 = \frac{-\nu}{E}(\sigma_1 + \sigma_2). \tag{3.33}$$

The geometry of Figure 3–5 suggests that the plane stress formulas may be applicable to horizontal tectonic stresses in the lithosphere. Let us assume that in addition to the lithostatic stresses there are equal horizontal components of principal stress $\Delta\sigma_1 = \Delta\sigma_2$. According to Equations (3–31) to (3–33), the horizontal tectonic stresses produce the strains

$$\varepsilon_1 = \varepsilon_2 = \frac{(1-\nu)}{E}\Delta\sigma_1 \tag{3.34}$$

$$\varepsilon_3 = \frac{-2\nu}{E}\Delta\sigma_1. \tag{3.35}$$

If the horizontal tectonic stresses are compressive, vertical columns of lithosphere of initial thickness h_L, horizontal area A, and density ρ will undergo a decrease in area and an increase in thickness. The mass in a column will remain constant, however. Therefore we can write

$$\delta(\rho A\, h_L) = 0. \tag{3.36}$$

The weight per unit area at the base of the column $\rho g h_L$ will increase, as can be seen from

$$\begin{aligned}\delta(\rho g h_L) &= \delta\left(\rho g h_L A \cdot \frac{1}{A}\right)\\ &= \frac{1}{A}\delta(\rho g h_L A) + \rho g h_L A\ \delta\left(\frac{1}{A}\right)\\ &= \rho g h_L A\left(-\frac{1}{A^2}\right)\delta A = \rho g h_L\left(-\frac{\delta A}{A}\right).\end{aligned} \tag{3.37}$$

The term $\delta(\rho g h_L A)/A$ is zero from Equation (3–36); $\delta(\rho g h_L)$ is positive because $-\delta A/A$ is a positive quantity given by

$$-\frac{\delta A}{A} = \varepsilon_1 + \varepsilon_2 = \frac{2(1-\nu)}{E}\Delta\sigma_1. \tag{3.38}$$

The increase in the weight per unit area at the base of the lithospheric column gives the increase in the vertical principal stress $\Delta\sigma_3$. By combining Equations (3–37) and (3–38), we get

$$\Delta\sigma_3 = \frac{2(1-\nu)\rho g h_L}{E}\Delta\sigma_1 \tag{3.39}$$

or

$$\frac{\Delta\sigma_3}{\Delta\sigma_1} = \frac{2(1-\nu)\rho g h_L}{E}. \tag{3.40}$$

Taking $\nu = 0.25$, $E = 100$ GPa, $\rho = 3000$ kg m^{-3}, $g = 10$ m s^{-2}, and $h_L =$ 100 km as typical values for the lithosphere, we find that $\Delta\sigma_3/\Delta\sigma_1 = 0.045$. Because the change in the vertical principal stress is small compared with the applied horizontal principal stresses, we conclude that the plane stress assumption is valid for the earth's lithosphere.

Problem 3.5 Triaxial compression tests are a common laboratory technique for determining elastic properties and strengths of rocks at various pressures p and temperatures. Figure 3–6 is a schematic of the experimental method. A cylindrical rock specimen is loaded axially by a compressive stress σ_1. The sample is also uniformly compressed laterally by stresses $\sigma_2 = \sigma_3 < \sigma_1$.

Show that

$$\varepsilon_2 = \varepsilon_3$$

and

$$\sigma_1 - \sigma_2 = 2G(\varepsilon_1 - \varepsilon_2).$$

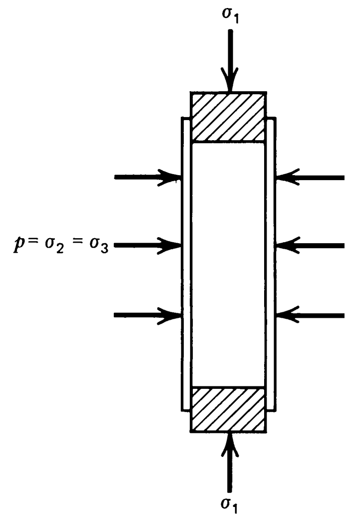

Figure 3.6 Sketch of a triaxial compression test on a cylindrical rock sample.

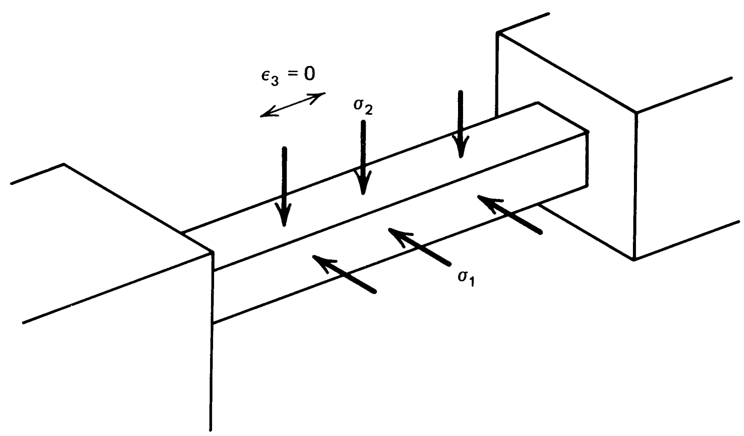

Figure 3.7 An example of plane strain.

Thus if the measured stress difference $\sigma_1 - \sigma_2$ is plotted against the measured strain difference $\varepsilon_1 - \varepsilon_2$, the slope of the line determines $2G$.

3.6 Plane Strain

In the case of *plane strain*, $\varepsilon_3 = 0$, for example, and ε_1 and ε_2 are nonzero. Figure 3–7 illustrates a plane strain situation. A long bar is rigidly confined between supports so that it cannot expand or contract parallel to its length.

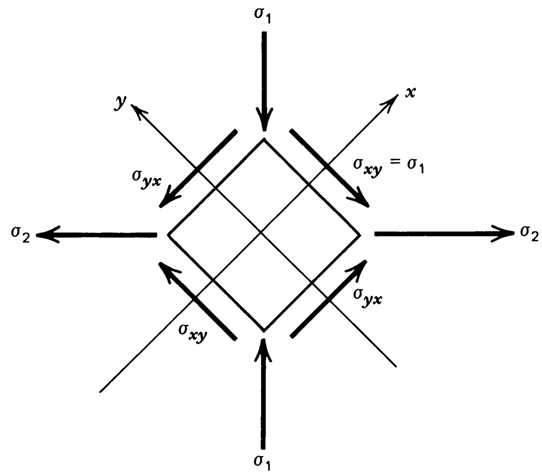

Figure 3.8 Principal stresses and shear stresses in the case of pure shear.

In addition, the stresses σ_1 and σ_2 are applied uniformly along the length of the bar.

Equations (3–1) to (3–3) reduce to

$$\sigma_1 = (\lambda + 2G)\varepsilon_1 + \lambda\varepsilon_2 \tag{3.41}$$

$$\sigma_2 = \lambda\varepsilon_1 + (\lambda + 2G)\varepsilon_2 \tag{3.42}$$

$$\sigma_3 = \lambda(\varepsilon_1 + \varepsilon_2). \tag{3.43}$$

From Equation (3–6) it is obvious that

$$\sigma_3 = \nu(\sigma_1 + \sigma_2). \tag{3.44}$$

This can be used together with Equations (3–4) and (3–5) to find

$$\varepsilon_1 = \frac{(1+\nu)}{E}\{\sigma_1(1-\nu) - \nu\sigma_2\} \tag{3.45}$$

$$\varepsilon_2 = \frac{(1+\nu)}{E}\{\sigma_2(1-\nu) - \nu\sigma_1\}. \tag{3.46}$$

3.7 Pure Shear and Simple Shear

The state of stress associated with pure shear is illustrated in Figure 3–8. Pure shear is a special case of plane stress. One example of pure shear is $\sigma_3 = 0$ and $\sigma_1 = -\sigma_2$. From Equations (2–56) to (2–58) with $\theta = -45°$ (compare Figures 2–14 and 3–8), we find that $\sigma_{xx} = \sigma_{yy} = 0$ and $\sigma_{xy} = \sigma_1$. In this coordinate system only the shear stress is nonzero. From Equations

(3–31) and (3–32) we find that

$$\varepsilon_1 = \frac{(1+\nu)}{E}\sigma_1 = \frac{(1+\nu)}{E}\sigma_{xy} = -\varepsilon_2, \tag{3.47}$$

and from Equations (2–130) and (2–131) with $\theta = -45^\circ$ we get $\varepsilon_{xx} = \varepsilon_{yy} = 0$ and $\varepsilon_{xy} = \varepsilon_1$. Equation (3–47) then gives

$$\sigma_{xy} = \frac{E}{1+\nu}\varepsilon_{xy}. \tag{3.48}$$

By introducing the modulus of rigidity from Equation (3–12), we can write the shear stress as

$$\sigma_{xy} = 2G\varepsilon_{xy}, \tag{3.49}$$

which explains why the modulus of rigidity is also known as the shear modulus. (Note: In terms of $\gamma_{xy} \equiv 2\varepsilon_{xy}, \sigma_{xy} = G\gamma_{xy}$.) These results are valid for both pure shear and simple shear because the two states differ by a solid-body rotation that does not affect the state of stress.

Simple shear is generally associated with displacements on a strike–slip fault such as the San Andreas in California. In Equation (2–134) we concluded that the shear strain associated with the 1906 San Francisco earthquake was 2.5×10^{-5}. With $G = 30$ GPa, Equation (3–49) gives the related shear stress as 1.5 MPa. This is a very small stress drop to be associated with a great earthquake. However, for the stress drop to have been larger, the width of the zone of strain accumulation would have had to have been even smaller. If the stress had been 15 MPa, the width of the zone of strain accumulation would have had to have been 4 km on each side of the fault. We will return to this problem in Chapter 8.

Problem 3.6 Show that Equation (3–49) can also be derived by assuming plane strain.

3.8 Isotropic Stress

If all the principal stresses are equal $\sigma_1 = \sigma_2 = \sigma_3 \equiv p$, then the state of stress is isotropic, and the principal stresses are equal to the pressure. The principal strains in a solid subjected to isotropic stresses are also equal $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \frac{1}{3}\Delta$; each component of strain is equal to one-third of the dilatation. By adding Equations (3–1) to (3–3), we find

$$p = \left(\frac{3\lambda + 2G}{3}\right)\Delta \equiv K\Delta \equiv \frac{1}{\beta}\Delta. \tag{3.50}$$

The quantity K is the *bulk modulus*, and its reciprocal is β, the *compressibility*. The ratio of p to the bulk modulus gives the fractional volume change that occurs under isotropic compression.

Because the mass of a solid element with volume V and density ρ must be conserved, any change in volume δV of the element must be accompanied by a change in its density $\delta\rho$. The fractional change in density can be related to the fractional change in volume, the dilatation, by rearranging the equation of mass conservation

$$\delta(\rho V) = 0, \tag{3.51}$$

which gives

$$\rho\delta V + V\delta\rho = 0 \tag{3.52}$$

or

$$\frac{-\delta V}{V} = \Delta = \frac{\delta\rho}{\rho}. \tag{3.53}$$

Equation (3–53) of course assumes Δ to be small. The combination of Equations (3–50) and (3–53) gives

$$\delta\rho = \rho\beta p. \tag{3.54}$$

This relationship can be used to determine the increase in density with depth in the earth.

Using Equations (3–11) to (3–13), we can rewrite the formula for K given in Equation (3–50) as

$$K = \frac{1}{\beta} = \frac{E}{3(1-2\nu)}. \tag{3.55}$$

Thus as ν tends toward 1/2, that is, as a material becomes more and more incompressible, its bulk modulus tends to infinity.

3.9 Two-Dimensional Bending or Flexure of Plates

We have already discussed how plate tectonics implies that the near-surface rocks are rigid and therefore behave elastically on geological time scales. The thin elastic surface plates constitute the lithosphere, which floats on the relatively fluid mantle beneath. The plates are subject to a variety of loads – volcanoes, seamounts, for example – that force the lithosphere to bend under their weights. By relating the observed *flexure* or *bending* of the lithosphere to known surface loads, we can deduce the elastic properties and thicknesses of the plates. In what follows, we first develop the theory of

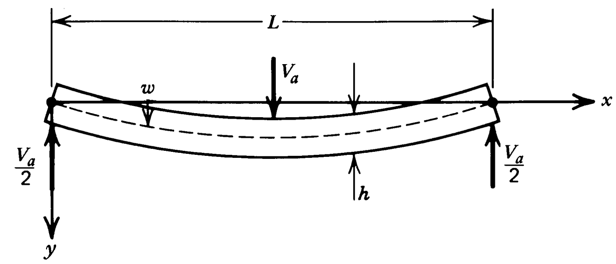

Figure 3.9 A thin plate of length L and thickness h pinned at its ends and bending under and applied load V_a.

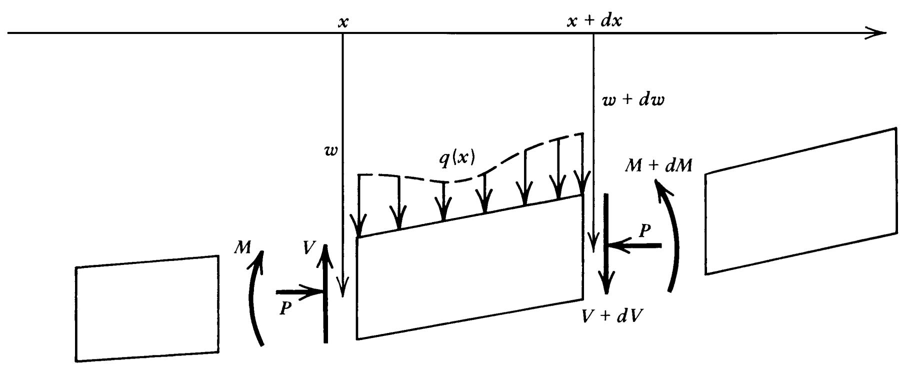

Figure 3.10 Forces and torques on a small section of a deflecting plate.

plate bending in response to applied forces and torques. The theory can also be used to understand fold trains in mountain belts by modeling the folds as deformations of elastic plates subject to horizontal compressive forces. Other geologic applications also can be made. For example, we will apply the theory to model the upwarping of strata overlying igneous intrusions (Section 3–12).

A simple example of plate bending is shown in Figure 3–9. A plate of thickness h and width L is pinned at its ends and bends under the load of a line force V_a (N m^{-1}) applied at its center. The plate is infinitely long in the z direction. A vertical, static force balance and the symmetry of the situation require that equal vertical line forces $V_a/2$ be applied at the supports. The plate is assumed to be thin compared with its width, $h \ll L$, and the vertical deflection of the plate w is taken to be small, $w \ll L$. The latter assumption is necessary to justify the use of linear elastic theory. The two-dimensional bending of plates is also referred to as *cylindrical bending* because the plate takes the form of a segment of a cylinder.

The deflection of a plate can be determined by requiring it to be in equilibrium under the action of all the forces and torques exerted on it. The forces and torques on a small section of the plate between horizontal locations x and $x + dx$ are shown in Figure 3–10. A downward force per unit area $q(x)$ is exerted on the plate by whatever distributed load the plate is required to support. Thus, the downward load, per unit length in the z direction, between x and $x + dx$ is $q(x)\,dx$. A net shear force V, per unit length in the z direction, acts on the cross section of the plate normal to the plane of the figure; it is the resultant of all the shear stresses integrated over that cross-sectional area of the plate. A horizontal force P, per unit length in the z direction, is applied to the plate; it is assumed that P is independent of x. The net *bending moment* M, per unit length in the z direction, acting on a cross section of the plate is the integrated effect of the moments exerted by the normal stresses σ_{xx}, also known as the *fiber stresses*, on the cross section. We relate M to the fiber stresses in the plate later in the discussion. All quantities in Figure 3–10 are considered positive when they have the sense shown in the figure. At location x along the plate the shear force is V, the bending moment is M, and the deflection is w; at $x+dx$, the shear force is $V + dV$, the bending moment is $M + dM$, and the deflection is $w + dw$. It is to be emphasized that V, M, and P are per unit length in the z direction.

A force balance in the vertical direction on the element between x and $x + dx$ yields

$$q(x)\,dx + dV = 0 \tag{3.56}$$

or

$$\frac{dV}{dx} = -q. \tag{3.57}$$

The moments M and $M+dM$ combine to give a net counterclockwise torque dM on the element. The forces V and $V+dV$ are separated by a distance dx (an infinitesimal *moment arm*) and exert a net torque $V dx$ on the element in a clockwise sense. (The change in V in going from x to $x + dx$ can be ignored in calculating the moment due to the shear forces.) The horizontal forces P exert a net counterclockwise torque $-P\,dw$ on the element through their associated moment arm $-dw$. (Note that dw is negative in going from x to $x + dx$.) A balance of all the torques gives

$$dM - P\,dw = V\,dx \tag{3.58}$$

or

$$\frac{dM}{dx} = V + P\frac{dw}{dx}. \tag{3.59}$$

We can eliminate the shear force on a vertical cross section of the plate V

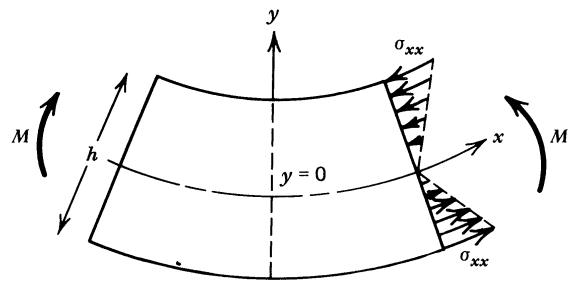

Figure 3.11 The normal stresses on a cross section of a thin curved elastic plate.

from Equation (3–59) by differentiating the equation with respect to x and substituting from Equation (3–57). One obtains

$$\frac{d^2M}{dx^2} = -q + P\frac{d^2w}{dx^2}. \tag{3.60}$$

Equation (3–60) can be converted into a differential equation for the deflection w if the bending moment M can be related to the deflection; we will see that M is inversely proportional to the *local radius of curvature of the plate* R and that R^{-1} is $-d^2w/dx^2$.

To relate M to the curvature of the plate, we proceed as follows. If the plate is deflected downward, as in Figure 3–11, the upper half of the plate is contracted, and the longitudinal stress σ_{xx} is positive; the lower part of the plate is extended, and σ_{xx} is negative. The fiber stress σ_{xx} is zero on the midplane $y = 0$, which is a *neutral unstrained surface*. The net effect of these stresses is to exert a counterclockwise bending moment on the cross section of the plate. The curvature of the plate has, of course, been exaggerated in Figure 3–11 so that x is essentially horizontal. The force on an element of the plate's cross section of thickness dy is $\sigma_{xx}\,dy$. This force exerts a torque about the midpoint of the plate given by $\sigma_{xx}y\,dy$. If we integrate this torque over the cross section of the plate, we obtain the bending moment

$$M = \int_{-h/2}^{h/2} \sigma_{xx}y\,dy, \tag{3.61}$$

where h is the thickness of the plate.

The bending stress σ_{xx} is accompanied by longitudinal strain ε_{xx} that is positive (contraction) in the upper half of the plate and negative (extension) in the lower half. There is no strain in the direction perpendicular to the xy plane because the plate is infinite in this direction and *the bending is two-dimensional or cylindrical*; that is, $\varepsilon_{zz} = 0$. There is also zero stress normal to the surface of the plate; that is, $\sigma_{yy} = 0$. Because the plate is thin, we can take $\sigma_{yy} = 0$ throughout. Thus plate bending is an example of plane stress, and we can use Equations (3–31) and (3–32) to relate the stresses and strains; that is,

$$\varepsilon_{xx} = \frac{1}{E}(\sigma_{xx} - \nu\sigma_{zz}) \tag{3.62}$$

$$\varepsilon_{zz} = \frac{1}{E}(\sigma_{zz} - \nu\sigma_{xx}). \tag{3.63}$$

In writing these equations, we have identified the principal strains ε_1, ε_2 with ε_{xx}, ε_{zz} and the principal stresses σ_1, σ_2 with σ_{xx}, σ_{zz}. With $\varepsilon_{zz} = 0$, Equations (3–62) and (3–63) give

$$\sigma_{xx} = \frac{E}{(1-\nu^2)}\varepsilon_{xx}. \tag{3.64}$$

Equation (3–61) for the bending moment can be rewritten, using Equation (3–64), as

$$M = \frac{E}{(1-\nu^2)}\int_{-h/2}^{h/2} \varepsilon_{xx} y \, dy. \tag{3.65}$$

The longitudinal strain ε_{xx} depends on the distance from the midplane of the plate y and the local radius of curvature of the plate R. Figure 3–12 shows a bent section of the plate originally of length l (l is infinitesimal). The length of the section measured along the midplane remains l. The small angle ϕ is l/R in radians. The geometry of Figure 3–12 shows that the change in length of the section Δl at a distance y from the midplane is

$$\Delta l = -y\phi = -y\frac{l}{R}, \tag{3.66}$$

where the minus sign is included because there is contraction when y is positive. Thus the strain is

$$\varepsilon_{xx} = -\frac{\Delta l}{l} = \frac{y}{R}. \tag{3.67}$$

Implicit in this relation is the assumption that plane sections of the plate remain plane.

The local radius of curvature R is determined by the change in slope of the plate midplane with horizontal distance. The geometry is shown in Figure 3–13. If w is small, $-dw/dx$, the slope of the midplane, is also the angular deflection of the plate from the horizontal α. The small angle ϕ in Figure 3–13 is simply the change in α, that is, $d\alpha$, in the small distance l or dx.

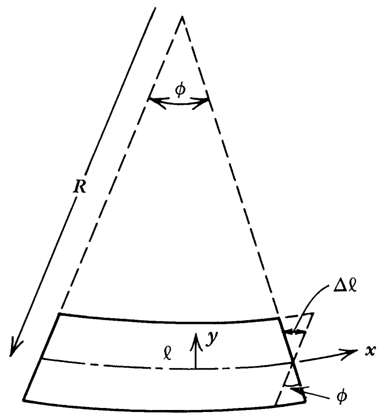

Figure 3.12 Longitudinal extension and contraction at a distance y from the midplane of the plate.

Thus

$$\phi = d\alpha = \frac{d\alpha}{dx}dx = \frac{d}{dx}\left(-\frac{dw}{dx}\right)dx = -\frac{d^2w}{dx^2}\,dx, \tag{3.68}$$

and we find

$$\frac{1}{R} = \frac{\phi}{l} \approx \frac{\phi}{dx} = -\frac{d^2w}{dx^2}. \tag{3.69}$$

Finally, the strain is given by

$$\varepsilon_{xx} = -y\frac{d^2w}{dx^2}, \tag{3.70}$$

and the bending moment can be written

$$\begin{aligned} M &= \frac{-E}{(1-\nu^2)}\frac{d^2w}{dx^2}\int_{-h/2}^{h/2} y^2\,dy \\ &= \frac{-E}{(1-\nu^2)}\frac{d^2w}{dx^2}\left(\frac{y^3}{3}\right)_{-h/2}^{h/2} \\ &= \frac{-Eh^3}{12(1-\nu^2)}\frac{d^2w}{dx^2}. \end{aligned} \tag{3.71}$$

The coefficient of $-d^2w/dx^2$ on the right side of Equation (3–71) is called the *flexural rigidity* D of the plate

$$D \equiv \frac{Eh^3}{12(1-\nu^2)}. \tag{3.72}$$

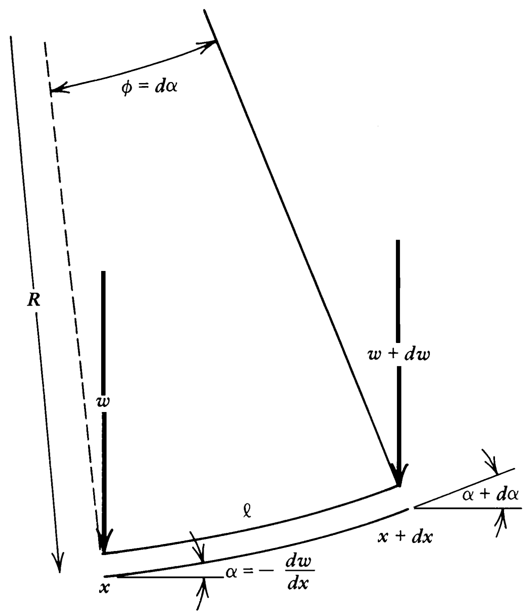

Figure 3.13 Sketch illustrating the geometrical relations in plate bending.

According to Equations (3–69), (3–71), and (3–72), the bending moment is the flexural rigidity of the plate divided by its curvature

$$M = -D\frac{d^2w}{dx^2} = \frac{D}{R}. \tag{3.73}$$

Upon substituting the second derivative of Equation (3–73) into Equation (3–60), we obtain the general equation for the deflection of the plate

$$D\frac{d^4w}{dx^4} = q(x) - P\frac{d^2w}{dx^2}. \tag{3.74}$$

We next solve Equation (3–74) for plate deflection in a number of simple cases and apply the results to the deformation of crustal strata and to the bending of the lithosphere.

3.10 Bending of Plates under Applied Moments and Vertical Loads

Consider a plate *embedded* at one end and subject to an applied torque M_a at the other, as shown in Figure 3–14. Assume for simplicity that the plate is weightless. With $q = 0$, Equation (3–57) shows that the shear stress on a section of the plate V must be a constant. In fact, $V = 0$, since there is no applied force acting on the plate. This can easily be seen by considering

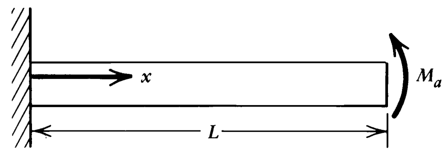

Figure 3.14 An embedded plate subject to an applied torque.

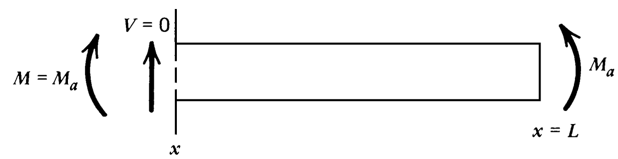

Figure 3.15 Force and torque balance on a section of the plate in Figure 3–14.

a force balance on a section of the plate, as shown in Figure 3–15. Since $P = 0$ and since we have established $V = 0$, Equation (3–59) requires that $M =$ constant. The constant must be M_a, the applied torque, as shown by a *moment balance* on an arbitrary section of the plate (Figure 3–15).

To determine the deflection of the plate, we could integrate Equation (3–74) with $q = P = 0$. However, since we already know $M \equiv M_a$, it is simpler to integrate Equation (3–73), the twice integrated form of the fourth-order differential equation. The boundary conditions are $w = 0$ at $x = 0$ and $dw/dx = 0$ at $x = 0$. These boundary conditions at the left end of the plate clarify what is meant by an embedded plate; the embedded end of the plate cannot be displaced, and its slope must be zero. The integral of Equation (3–73) subject to these boundary conditions is

$$w = \frac{-M_a x^2}{2D}. \tag{3.75}$$

The bent plate has the shape of a parabola. w is negative according to the convention we established if M is positive; that is, the plate is deflected upward.

Problem 3.7 What is the displacement of a plate pinned at both ends ($w = 0$ at $x = 0, L$) with equal and opposite bending moments applied at the ends? The problem is illustrated in Figure 3–16.

As a second example we consider the bending of a plate embedded at its left end and subjected to a concentrated force V_a at its right end, as illustrated in Figure 3–17. In this situation, $q = 0$, except at the point $x = L$, and Equation (3–57) gives $V =$ constant. The constant must be

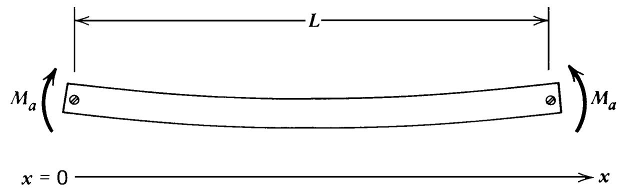

Figure 3.16 Bending of a plate pinned at both ends.

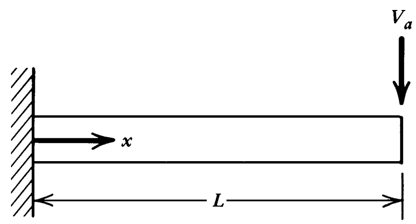

Figure 3.17 An embedded plate subjected to a concentrated load.

V_a, as shown by the vertical force balance on the plate sketched in Figure 3–18. With P also equal to zero, Equation (3–59) for the bending moment simplifies to

$$\frac{dM}{dx} = V_a. \tag{3.76}$$

This equation can be integrated to yield

$$M = V_a x + \text{constant}, \tag{3.77}$$

and the constant can be evaluated by noting that there is no applied torque at the end $x = L$; that is, $M = 0$ at $x = L$. Thus we obtain

$$M = V_a(x - L). \tag{3.78}$$

The bending moment changes linearly from $-V_a L$ at the embedded end to zero at the free end. A simple torque balance on the section of the plate shown in Figure 3–18 leads to Equation (3–78), since M must balance the torque of the applied force V_a acting with moment arm $L - x$.

The displacement can be determined by integrating Equation (3–74), which simplifies to

$$\frac{d^4 w}{dx^4} = 0, \tag{3.79}$$

when $q = P = 0$. The integral of Equation (3–79) is

$$\frac{d^3 w}{dx^3} = \text{constant}. \tag{3.80}$$

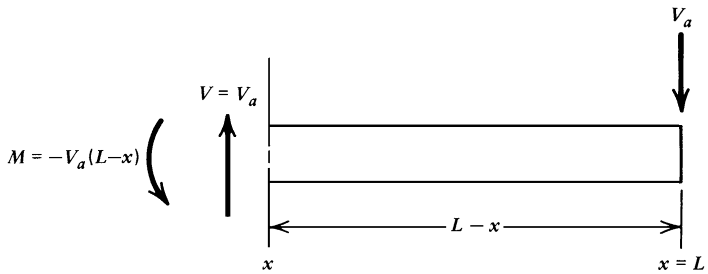

Figure 3.18 Forces and torques on a section of a plate loaded at its right end by a force V_a.

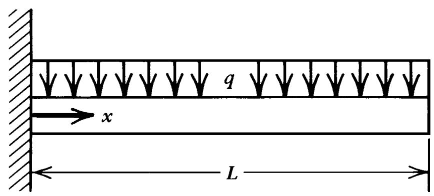

Figure 3.19 A uniformly loaded plate embedded at one end.

The constant can be evaluated by differentiating Equation (3–73) with respect to x and substituting for dM/dx from Equation (3–76). The result is

$$\frac{d^3w}{dx^3} = -\frac{V_a}{D}. \tag{3.81}$$

A second-order differential equation for w can be obtained by integrating Equation (3–81) and evaluating the constant of integration with the boundary condition $d^2w/dx^2 = 0$ at $x = L$. Alternatively, the same equation can be arrived at by substituting for M from Equation (3–78) into Equation (3–73)

$$\frac{d^2w}{dx^2} = -\frac{V_a}{D}(x - L). \tag{3.82}$$

This equation may be integrated twice more subject to the standard boundary conditions $w = dw/dx = 0$ at $x = 0$. One finds

$$w = \frac{V_a x^2}{2D}\left(L - \frac{x}{3}\right). \tag{3.83}$$

Problem 3.8 Determine the displacement of a plate of length L pinned at its ends with a concentrated load V_a applied at its center. This problem is illustrated in Figure 3–9.

As a third and final example, we consider the bending of a plate embedded at one end and subjected to a uniform loading $q(x) = \text{constant}$, as illustrated in Figure 3–19. Equation (3–74), with $P = 0$, becomes

$$\frac{d^4w}{dx^4} = \frac{q}{D}. \tag{3.84}$$

We need four boundary conditions to integrate Equation (3–84). Two of them are the standard conditions $w = dw/dx = 0$ at the left end $x = 0$. A third boundary condition is the same as the one used in the previous example, namely, $d^2w/dx^2 = 0$ at $x = L$, because there is no external torque applied at the right end of the plate – see Equation (3–73). The fourth boundary condition follows from Equation (3–59) with $P = 0$. Because there is no applied concentrated load at $x = L$, V must vanish there, as must dM/dx and from Equation (3–73), d^3w/dx^3. After some algebra, one finds the solution

$$w = \frac{qx^2}{D}\left(\frac{x^2}{24} - \frac{Lx}{6} + \frac{L^2}{4}\right). \tag{3.85}$$

The shear force at $x = 0$ is $-D(d^3w/dx^3)_{x=0}$. From Equation (3–85) this is qL, a result that also follows from a consideration of the overall vertical equilibrium of the plate because qL is the total loading. The shear stress on the section $x = 0$ is qL/h. The bending moment on the section $x = 0$ is $-D(d^2w/dx^2)_{x=0}$ or $-qL^2/2$. The maximum bending or fiber stress, $\sigma_{xx}^{\max} = \sigma_{xx}$ at $y = -h/2$, is given, from Equations (3–85), (3–64), and (3–70), by

$$\sigma_{xx}^{\max} = \frac{E}{(1-\nu^2)}\frac{h}{2}\frac{d^2w}{dx^2} = \frac{6}{h^2}D\frac{d^2w}{dx^2} = -\frac{6M}{h^2}. \tag{3.86}$$

At $x = 0$, $\sigma_{xx}^{\max}$ is $3qL^2/h^2$. The ratio of the shear stress to the maximum bending stress at $x = 0$ is $h/3L$, a rather small quantity for a thin plate. It is implicit in the analysis of the bending of thin plates that shear stresses in the plates are small compared with the bending stresses.

Problem 3.9 Calculate V and M by carrying out force and torque balances on the section of the uniformly loaded plate shown in Figure 3–20.

Problem 3.10 A granite plate with $\rho = 2700$ kg m^{-3} is embedded at one end. If $L = 10$ m and $h = 1/4$ m, what is the maximum bending stress and the shear stress at the base?

Problem 3.11 Determine the displacement of a plate that is embedded at the end $x = 0$ and has a uniform loading q from $x = L/2$ to $x = L$.

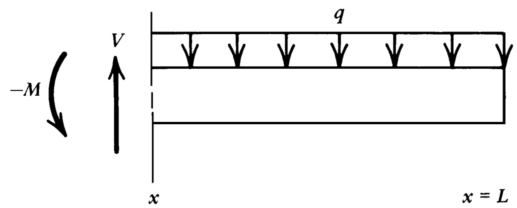

Figure 3.20 Section of a uniformly loaded plate.

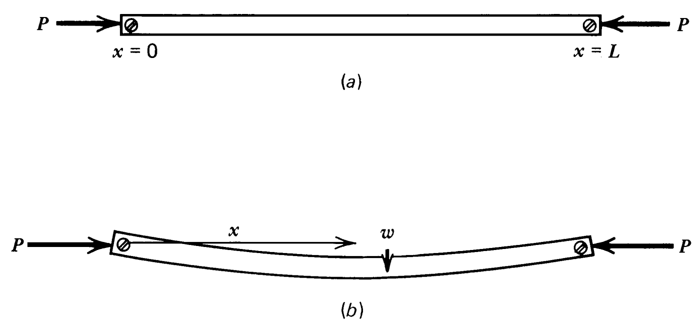

Figure 3.21 Plate buckling under a horizontal force.

Problem 3.12 Determine the deflection of a plate of length L that is embedded at $x = 0$ and has equal loads V_a applied at $x = L/2$ and at $x = L$.

Problem 3.13 Find the deflection of a uniformly loaded beam pinned at the ends, $x = 0, L$. Where is the maximum bending moment? What is the maximum bending stress?

Problem 3.14 A granite plate freely supported at its ends spans a gorge 20 m wide. How thick does the plate have to be if granite fails in tension at 20 MPa? Assume $\rho = 2700$ kg m^{-3}.

Problem 3.15 Determine the deflection of a freely supported plate, that is, a plate pinned at its ends, of length L and flexural rigidity D subject to a sinusoidal load $q_a = q_0 \sin \pi x/L$, as shown in Figure 3–21.

3.11 Buckling of a Plate under a Horizontal Load

When an elastic plate is subjected to a horizontal force P, as shown in Figure 3–22*a*, the plate can *buckle*, as illustrated in Figure 3–22*b*, if the applied force is sufficiently large. Fold trains in mountain belts are believed to result from the warping of strata under horizontal compression. We will therefore

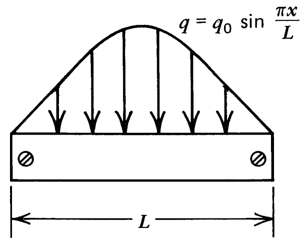

Figure 3.22 A freely supported plate loaded sinusoidally.

consider the simplest example of plate buckling under horizontal compression to determine the minimum force required for buckling to occur and the form, that is, the *wavelength*, of the resulting deflection. In a subsequent section we will carry out a similar calculation to determine if the lithosphere can be expected to buckle under horizontal tectonic compression.

We consider a plate pinned at both ends and subjected to a horizontal force P, as shown in Figure 3–22. The deflection of the plate is governed by Equation (3–74) with $q = 0$:

$$D\frac{d^4w}{dx^4} + P\frac{d^2w}{dx^2} = 0. \tag{3.87}$$

This can be integrated twice to give

$$D\frac{d^2w}{dx^2} + Pw = c_1x + c_2. \tag{3.88}$$

However, we require that w is zero at $x = 0,\ L$ and that $d^2w/dx^2 = 0$ at $x = 0,\ L$, since there are no applied torques at the ends. These boundary conditions require that $c_1 = c_2 = 0$, and Equation (3–88) reduces to

$$D\frac{d^2w}{dx^2} + Pw = 0. \tag{3.89}$$

Equation (3–89) has the general solution

$$w = c_1 \sin\left(\frac{P}{D}\right)^{1/2} x + c_2 \cos\left(\frac{P}{D}\right)^{1/2} x, \tag{3.90}$$

where c_1 and c_2 are constants of integration. Because w is equal to zero at $x = 0$, c_2 must be zero, and

$$w = c_1 \sin\left(\frac{P}{D}\right)^{1/2} x. \tag{3.91}$$

But w must also vanish at $x = L$, which implies that if $c_1 \neq 0$, then

$$\sin\left(\frac{P}{D}\right)^{1/2} L = 0. \tag{3.92}$$

Thus $(P/D)^{1/2}L$ must be an integer multiple of π,

$$\left(\frac{P}{D}\right)^{1/2} L = n\pi \qquad n = 1, 2, 3, \ldots \tag{3.93}$$

Solving this equation for P, we get

$$P = \frac{n^2\pi^2}{L^2} D. \tag{3.94}$$

Equation (3–94) defines a series of values of P for which nonzero solutions for w exist. The smallest such value is for $n = 1$ when P is given by

$$P = P_c = \frac{\pi^2}{L^2} D. \tag{3.95}$$

This is the minimum buckling load for the plate. If P is smaller than this *critical value*, known as an *eigenvalue*, the plate will not deflect under the applied load; that is, $c_1 = 0$ or $w = 0$. When P has the value given by Equation (3–95), the plate buckles or deflects under the horizontal load. At the onset of deflection the plate assumes the shape of a half sine curve

$$\begin{aligned} w &= c_1 \sin\left(\frac{P}{D}\right)^{1/2} x \\ &= c_1 \sin\frac{\pi x}{L}. \end{aligned} \tag{3.96}$$

The amplitude of the deflection cannot be determined by the linear analysis carried out here. Nonlinear effects fix the magnitude of the deformation.

The application of plate flexure theory to fold trains in mountain belts requires somewhat more complex models than considered here. Although a number of effects must be incorporated to approximate reality more closely, one of the most important is the influence of the medium surrounding a folded stratum. The rocks above and below a folded layer exert forces on the layer that influence the form (wavelength) of the folds and the critical horizontal force necessary to initiate buckling.

3.12 Deformation of Strata Overlying an Igneous Intrusion

A *laccolith* is a sill-like igneous intrusion in the form of a round lens-shaped body much wider than it is thick. Laccoliths are formed by magma that is intruded along bedding planes of flat, layered rocks at pressures so high that the magma raises the overburden and deforms it into a domelike shape. If the flow of magma is along a crack, a two-dimensional laccolith can be formed. Our analysis is restricted to this case. A photograph of a laccolithic mountain is given in Figure 3–23 along with a sketch of our model.

The overburden or elastic plate of thickness h is bent upward by the pressure p of the magma that will form the laccolith upon solidification. The loading of the plate $q(x)$ is the part of the upward pressure force p in excess of the lithostatic pressure ρgh:

$$q = -p + \rho gh. \tag{3.97}$$

This problem is very similar to the one illustrated in Figure 3–19. In both cases the loading is uniform so that Equation (3–84) is applicable. We take $x = 0$ at the center of the laccolith. The required boundary conditions are $w = dw/dx = 0$ at $x = \pm L/2$. The solution of Equation (3–84) that satisfies these boundary conditions is obtained after some algebra in the form

$$w = -\frac{(p - \rho gh)}{24D}\left(x^4 - \frac{L^2x^2}{2} + \frac{L^4}{16}\right). \tag{3.98}$$

Note that because of the symmetry of the problem the coefficients of x and x^3 must be zero. The maximum deflection at the center of the laccolith, $x = 0$, is

$$w_0 = -\frac{(p - \rho gh)L^4}{384D}. \tag{3.99}$$

In terms of its maximum value, the deflection is given by

$$w = w_0\left(1 - 8\frac{x^2}{L^2} + 16\frac{x^4}{L^4}\right). \tag{3.100}$$

Problem 3.16 Show that the cross-sectional area of a two-dimensional laccolith is given by $(p - \rho gh)L^5/720D$.

Problem 3.17 Determine the bending moment in the overburden above the idealized two-dimensional laccolith as a function of x. Where is M a maximum? What is the value of $M_{\max}$?

Problem 3.18 Calculate the fiber stress in the stratum overlying the two-dimensional laccolith as a function of y (distance from the centerline of the layer) and x. If dikes tend to form where tension is greatest in the base of the stratum forming the roof of a laccolith, where would you expect dikes to occur for the two-dimensional laccolith?

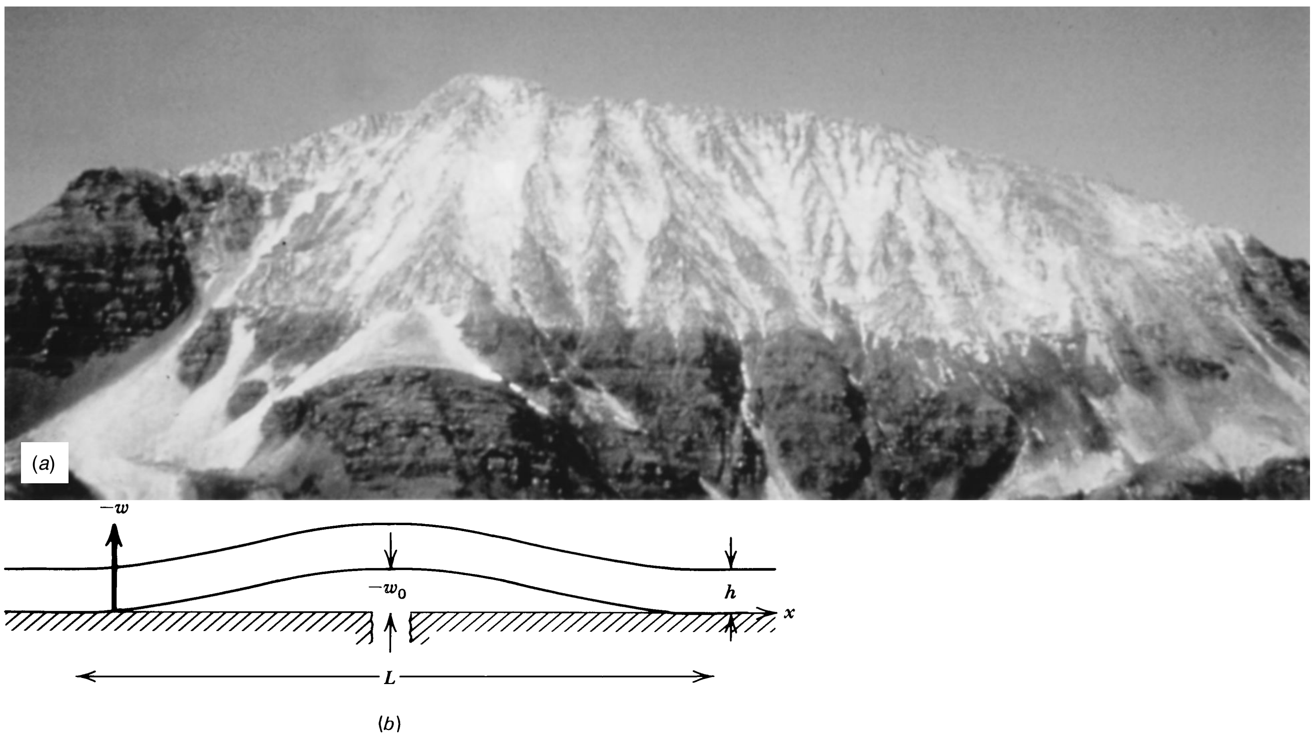

Figure 3.23 *(a) A laccolith in Red and White Mountain, Colorado. The overlying sedimentary rocks have been eroded (University of Colorado, Boulder). (b) A two-dimensional model for a laccolith.*

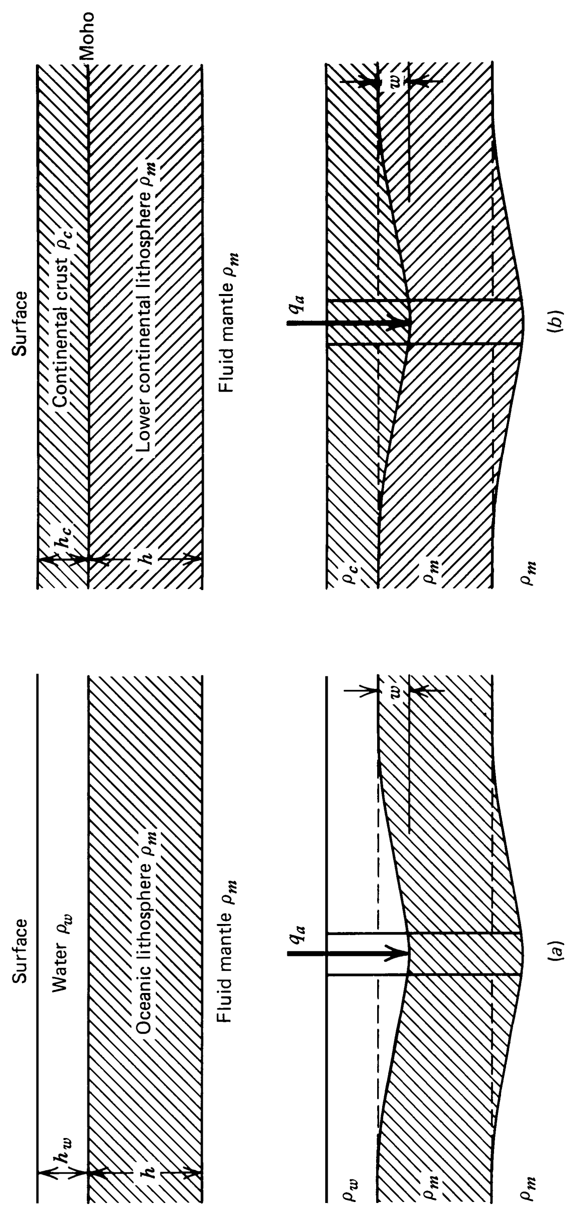

Figure 3.24 *Models for calculating the hydrostatic restoring force on lithospheric plates deflected by an applied load q_a. (a) Oceanic case. (b) Continental case.*

3.13 Application to the Earth's Lithosphere

When applying Equation (3–74) to determine the downward deflection of the earth's lithosphere due to an applied load, we must be careful to include in $q(x)$ the hydrostatic restoring force caused by the effective replacement of mantle rocks in a vertical column by material of smaller density. In the case of the oceanic lithosphere, water fills in "the space vacated" by mantle rocks moved out of the way by the deflected lithosphere. In the case of the continental lithosphere, the rocks of the thick continental crust serve as the fill. Figure 3–24*a* illustrates the oceanic case. The upper part of the figure shows a lithospheric plate of thickness h and density ρ_m floating on a "fluid" mantle also of density ρ_m. Water of density ρ_w and thickness h_w overlies the oceanic lithosphere. Suppose that an applied load deflects the lithosphere downward a distance w and that water fills in the space above the plate, as shown in the bottom part of Figure 3–24*a*. The weight per unit area of a vertical column extending from the base of the deflected lithosphere to the surface is

$$\rho_w g(h_w + w) + \rho_m gh.$$

The pressure at a depth $h_w + h + w$ in the surrounding mantle where there is no plate deflection is

$$\rho_w gh_w + \rho_m g(h + w).$$

Thus there is an upward hydrostatic force per unit area equal to

$$\begin{aligned} &\rho_w gh_w + \rho_m g(h + w) - \{\rho_w g(h_w + w) + \rho_m gh\} \\ &\quad = (\rho_m - \rho_w)gw \end{aligned} \tag{3.101}$$

tending to restore the deflected lithosphere to its original configuration. The hydrostatic restoring force per unit area is equivalent to the force that results from replacing mantle rock of thickness w and density ρ_m by water of thickness w and density ρ_w. The net force per unit area acting on the lithospheric plate is therefore

$$q = q_a - (\rho_m - \rho_w)gw, \tag{3.102}$$

where q_a is the applied load at the upper surface of the lithosphere. Equation (3–74) for the deflection of the elastic oceanic lithosphere becomes

$$D\frac{d^4w}{dx^4} + P\frac{d^2w}{dx^2} + (\rho_m - \rho_w)gw = q_a(x). \tag{3.103}$$

Figure 3–24*b* illustrates the continental case. The upper part of the figure shows the continental crust of thickness h_c and density ρ_c separated by the

Moho from the rest of the lithosphere of thickness h and density ρ_m. The entire continental lithosphere lies on top of a fluid mantle of density ρ_m. The lower part of Figure 3–24*b* shows the plate deflected downward a distance w by an applied load such as excess topography. The Moho, being a part of the lithosphere, is also deflected downward a distance w. The space vacated by the deflected lithosphere is filled in by crustal rocks. The crust beneath the load is effectively thickened by the amount w by which the Moho is depressed. The weight per unit area of a vertical column extending from the base of the deflected plate to the surface is

$$\rho_c g(h_c + w) + \rho_m g h.$$

The pressure at a depth $h_c + h + w$ in the surrounding mantle far from the deflected plate is

$$\rho_c g h_c + \rho_m g(h + w).$$

The difference between these two quantities is the upward hydrostatic restoring force per unit area

$$\begin{aligned} &\rho_c g h_c + \rho_m g(h + w) - \{\rho_c g(h_c + w) + \rho_m g h\} \\ &\quad = (\rho_m - \rho_c) g w. \end{aligned} \tag{3.104}$$

The restoring force is equivalent to the force that results from replacing mantle rock by crustal rock in a layer of thickness w. The net force per unit area acting on the elastic continental lithosphere is therefore

$$q = q_a - (\rho_m - \rho_c) g w. \tag{3.105}$$

Equation (3–74) for the deflection of the plate becomes

$$D\frac{d^4 w}{dx^4} + P\frac{d^2 w}{dx^2} + (\rho_m - \rho_c) g w = q_a(x). \tag{3.106}$$

We are now in a position to determine the elastic deflection of the lithosphere and the accompanying internal stresses (shear and bending) for different loading situations.

3.14 Periodic Loading

How does the positive load of a mountain or the negative load of a valley deflect the lithosphere? To answer this question, we determine the response of the lithosphere to a periodic load. We assume that the elevation of the topography is given by

$$h = h_0 \sin 2\pi\frac{x}{\lambda}, \tag{3.107}$$

where h is the topographic height and λ is its wavelength. Positive h corresponds to ridges and negative h to valleys. Since the amplitude of the topography is small compared with the thickness of the elastic lithosphere, the influence of the topography on this thickness can be neglected. The load on the lithosphere corresponding to the topography given by Equation (3–107) is

$$q_a(x) = \rho_c g h_0 \sin 2\pi \frac{x}{\lambda} \tag{3.108}$$

where ρ_c is the density of the crustal rocks associated with the height variation. The equation for the deflection of the lithosphere is obtained by substituting this expression for $q_a(x)$ into Equation (3–106) and setting $P = 0$ to obtain

$$D\frac{d^4w}{dx^4} + (\rho_m - \rho_c)gw = \rho_c g h_0 \sin 2\pi \frac{x}{\lambda}. \tag{3.109}$$

Because the loading is periodic in x, the response or deflection of the lithosphere will also vary sinusoidally in x with the same wavelength as the topography. Thus we assume a solution of the form

$$w = w_0 \sin 2\pi \frac{x}{\lambda}. \tag{3.110}$$

By substituting Equation (3–110) into Equation (3–109), we determine the amplitude of the deflection of the lithosphere to be

$$w_0 = \frac{h_0}{\dfrac{\rho_m}{\rho_c} - 1 + \dfrac{D}{\rho_c g}\left(\dfrac{2\pi}{\lambda}\right)^4}. \tag{3.111}$$

The quantity $(D/\rho_c g)^{1/4}$ has the dimensions of a length. It is proportional to the natural wavelength for the flexure of the lithosphere.

If the wavelength of the topography is sufficiently short, that is, if

$$\lambda \ll 2\pi \left(\frac{D}{\rho_c g}\right)^{1/4}, \tag{3.112}$$

then the denominator of Equation (3–111) is much larger than unity, and

$$w_0 \ll h_0. \tag{3.113}$$

Short-wavelength topography causes virtually no deformation of the lithosphere. The lithosphere is infinitely rigid for loads of this scale. This case is illustrated in Figure 3–25*a*. If the wavelength of the topography is sufficiently long, that is, if

$$\lambda \gg 2\pi \left(\frac{D}{\rho_c g}\right)^{1/4}, \tag{3.114}$$

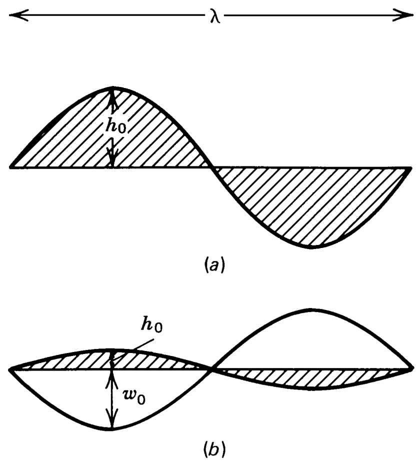

Figure 3.25 Deflection of the lithosphere under a periodic load. (*a*) Short-wavelength loading with no deflection of the lithosphere. (*b*) Long-wavelength loading with isostatic deflection of the lithosphere.

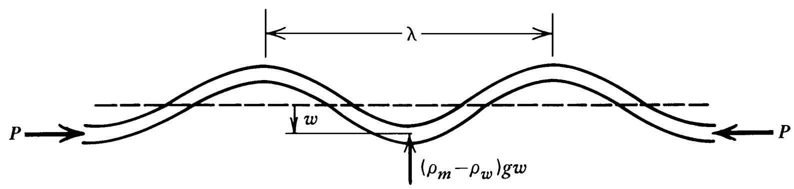

Figure 3.26 Buckling of an infinitely long plate under an applied horizontal load with a hydrostatic restoring force.

then Equation (3–111) gives

$$w = w_{0\infty} = \frac{\rho_c h_0}{(\rho_m - \rho_c)}. \tag{3.115}$$

This is the isostatic result obtained in Equation (2–3). For topography of sufficiently long wavelength, the lithosphere has no rigidity and the topography is fully compensated; that is, it is in hydrostatic equilibrium.

The degree of compensation C of the topographic load is the ratio of the deflection of the lithosphere to its maximum or hydrostatic deflection

$$C = \frac{w_0}{w_{0\infty}}. \tag{3.116}$$

Upon substituting Equations (3–111) and (3–115) into the equation for C, we obtain

$$C = \frac{(\rho_m - \rho_c)}{\rho_m - \rho_c + \dfrac{D}{g}\left(\dfrac{2\pi}{\lambda}\right)^4}. \tag{3.117}$$

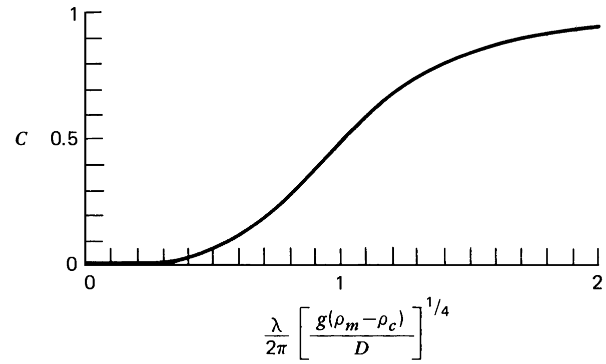

Figure 3.27 Dependence of the degree of compensation on the nondimensional wavelength of periodic topography.

This dependence is illustrated in Figure 3–26. For a lithosphere with elastic thickness 25 km, $E = 70$ GPa, $\nu = 0.25$, $\rho_m = 3300$ kg m^{-3}, and $\rho_c = 2800$ kg m^{-3} we find that topography is 50% compensated ($C = 0.5$) if its wavelength is $\lambda = 420$ km. Topography with a shorter wavelength is substantially supported by the rigidity of the lithosphere; topography with a longer wavelength is only weakly supported.

3.15 Stability of the Earth's Lithosphere under an End Load

We have already seen how a plate pinned at its ends can buckle if an applied horizontal load exceeds the critical value given by Equation (3–95). Let us investigate the stability of the lithosphere when it is subjected to a horizontal force P. We will see that when P exceeds a critical value, an infinitely long plate ($L \to \infty$) will become unstable and deflect into the sinusoidal shape shown in Figure 3–27.

The equation for the deflection of the plate is obtained by setting $q_a = 0$ in Equation (3–103):

$$D\frac{d^4w}{dx^4} + P\frac{d^2w}{dx^2} + (\rho_m - \rho_w)gw = 0. \tag{3.118}$$

This equation can be satisfied by a sinusoidal deflection of the plate as given in Equation (3–110) if

$$D\left(\frac{2\pi}{\lambda}\right)^4 - P\left(\frac{2\pi}{\lambda}\right)^2 + (\rho_m - \rho_w)g = 0, \tag{3.119}$$

a result of directly substituting Equation (3–110) into Equation (3–118). Equation (3–119) is a quadratic equation for the square of the wavelength of the sinusoid λ. Its solution is

$$\left(\frac{2\pi}{\lambda}\right)^2 = \frac{P \pm [P^2 - 4(\rho_m - \rho_w)gD]^{1/2}}{2D}. \tag{3.120}$$

Because the wavelength of the deformed lithosphere must be real, there can only be a solution if P exceeds the critical value

$$P_c = \{4Dg(\rho_m - \rho_w)\}^{1/2}. \tag{3.121}$$

P_c is the minimum value for P for which the initially horizontal lithosphere will become unstable and acquire the sinusoidal shape. If $P < P_c$, the horizontal lithosphere is stable and will not buckle under the end load.

The eigenvalue P_c can also be written

$$P_c = \left(\frac{Eh^3(\rho_m - \rho_w)g}{3(1-\nu^2)}\right)^{1/2} = \sigma_c h, \tag{3.122}$$

where σ_c is the critical stress associated with the force P_c. Solving Equation (3–122) for the critical stress we find

$$\sigma_c = \left(\frac{Eh(\rho_m - \rho_w)g}{3(1-\nu^2)}\right)^{1/2}. \tag{3.123}$$

The wavelength of the instability that occurs when $P = P_c$ is given by Equation (3–120):

$$\lambda_c = 2\pi\left(\frac{2D}{P_c}\right)^{1/2} = 2\pi\left(\frac{D}{g(\rho_m - \rho_w)}\right)^{1/4}$$
$$= 2\pi\left(\frac{Eh^3}{12(1-\nu^2)(\rho_m - \rho_w)g}\right)^{1/4}. \tag{3.124}$$

We wish to determine whether buckling of the lithosphere can lead to the formation of a series of synclines and anticlines. We consider an elastic lithosphere with a thickness of 50 km. Taking $E = 100$ GPa, $\nu = 0.25$, $\rho_m = 3300$ kg m^{-3}, and $\rho_w = 1000$ kg m^{-3}, we find from Equation (3–123) that $\sigma_c = 6.4$ GPa. A 50-km-thick elastic lithosphere can support a horizontal compressive stress of 6.4 GPa without buckling. Because of the very large stress required, we conclude that such buckling does not occur. The lithosphere fails, presumably by the development of a fault, before buckling can take place. In general, horizontal forces have a small influence on the bending behavior of the lithosphere. For this reason we neglect them in the lithosphere bending studies to follow.

Horizontal forces are generally inadequate to buckle the lithosphere because of its large elastic thickness. However, the same conclusion may not