5

Gravity

5.1 Introduction

The force exerted on an element of mass at the surface of the Earth has two principal components. One is due to the gravitational attraction of the mass in the Earth, and the other is due to the rotation of the Earth. Gravity refers to the combined effects of both gravitation and rotation. If the Earth were a nonrotating spherically symmetric body, the gravitational acceleration on its surface would be constant. However, because of the Earth's rotation, topography, and internal lateral density variations, the *acceleration of gravity* g varies with location on the surface. The Earth's rotation leads mainly to a latitude dependence of the surface acceleration of gravity. Because rotation distorts the surface by producing an *equatorial bulge* and a *polar flattening*, gravity at the equator is about 5 parts in 1000 less than gravity at the poles. The Earth takes the shape of an *oblate spheroid*. The gravitational field of this spheroid is the reference gravitational field of the Earth. Topography and density inhomogeneities in the Earth lead to local variations in the surface gravity, which are referred to as *gravity anomalies*.

The mass of the rock associated with topography leads to surface gravity anomalies. However, as we discussed in Chapter 2, large topographic features have low-density crustal roots. Just as the excess mass of the topography produces a positive gravity anomaly, the low-density root produces a negative gravity anomaly. In the mid-1800s it was observed that the gravitational attraction of the Himalayan Mountains was considerably less than would be expected because of the positive mass of the topography. This was the first evidence that the crust–mantle boundary is depressed under large mountain belts.

A dramatic example of the importance of crustal thickening is the absence of positive gravity anomalies over the continents. The positive mass anomaly associated with the elevation of the continents above the ocean floor is reduced or *compensated* by the negative mass anomaly associated with the thicker continental crust. We will show that compensation due to the hydrostatic equilibrium of thick crust leads in the first approximation to a zero value for the surface gravity anomaly. There are mechanisms for compensation other than the simple thickening of the crust. An example is the subsidence of the ocean floor due to the thickening of the thermal lithosphere, as discussed in Section 4–23.

Gravity anomalies that are correlated with topography can be used to study the flexure of the elastic lithosphere under loading. Short wavelength loads do not depress the lithosphere, but long wavelength loads result in flexure and a depression of the Moho. Gravity anomalies can also have important economic implications. Ore minerals are usually more dense than the country rock in which they are found. Therefore, economic mineral deposits are usually associated with positive gravity anomalies. Major petroleum occurrences are often found beneath salt domes. Since salt is less dense than other sedimentary rocks, salt domes are usually associated with negative gravity anomalies.

As we will see in the next chapter, mantle convection is driven by variations of density in the Earth's mantle. These variations produce gravity anomalies at the Earth's surface. Thus, measurements of gravity at the Earth's surface can provide important constraints on the flow patterns within the Earth's interior. However, it must be emphasized that the surface gravity does not provide a unique measure of the density distribution within the Earth's interior. Many different internal density distributions can give the same surface distributions of gravity anomalies. In other words, inversions of gravity data are non-unique.

5.2 Gravitational Acceleration External to the Rotationally Distorted Earth

The gravitational force exerted on a mass m' located at point P outside the Earth by a small element of mass dm in the Earth is given by *Newton's law of gravitation.* As shown in Figure 5–1, the gravitational attraction df_m in the direction from P to dm is given by

$$df_m = \frac{Gm'dm}{b^2}, \tag{5.1}$$

where G is the *universal gravitational constant* $G = 6.673 \times 10^{-11}$ m^3 kg^{-1} s^{-2} and b is the distance between dm and the point P. The infinitesimal

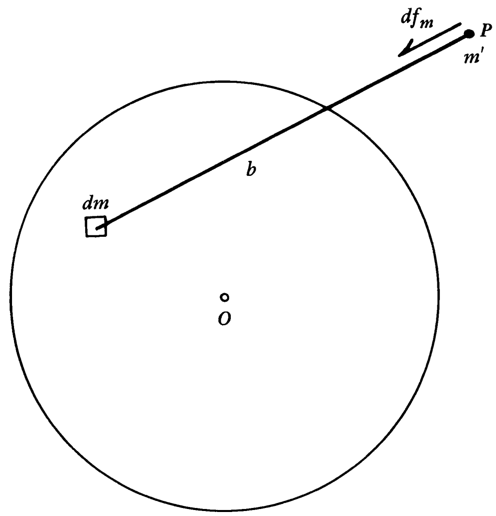

Figure 5.1 Force on a mass m' due to the gravitational attraction of an infinitesimal element of mass dm in the Earth.

gravitational acceleration at P due to the attraction of dm is the force per unit mass exerted on m' in the direction of P:

$$dg_m = \frac{df_m}{m'}. \tag{5.2}$$

By combining Equations (5–1) and (5–2) we obtain

$$dg_m = \frac{G\,dm}{b^2}. \tag{5.3}$$

If the distribution of mass in the Earth were known exactly, the gravitational attraction of the Earth on a unit mass outside the Earth could be obtained by summing or integrating dg_m over the entire distribution. Suppose, for example, that the entire mass of the Earth M were concentrated at its center. The gravitational acceleration at a distance r from the center would then be directed radially inward and, according to Equation (5–3), it would be given by

$$g_m = \frac{GM}{r^2}. \tag{5.4}$$

Following the generally accepted sign convention, we take g_m to be positive, even though it is directed in the $-r$ direction.

We next determine the gravitational acceleration outside a spherical body with a density distribution that is a function of radius only, $\rho = \rho(r')$. The geometry is illustrated in Figure 5–2. It is clear from symmetry considerations that the gravitational acceleration g_m at a point P outside the mass

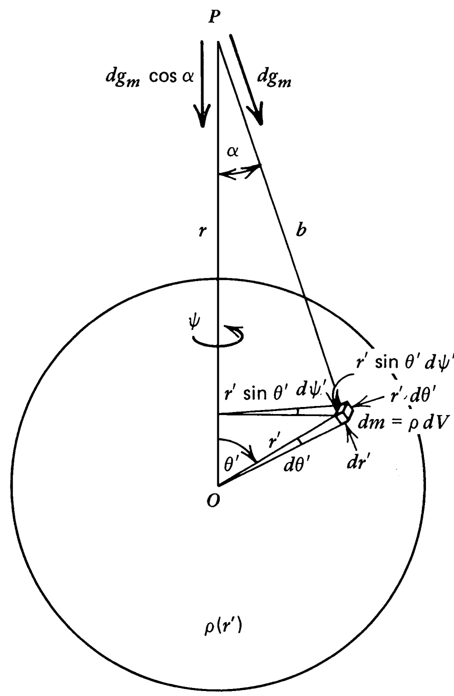

Figure 5.2 Geometry for the calculation of the gravitational acceleration at a point outside a spherically symmetric mass distribution.

distribution is directed radially inward and depends only on the distance r of point P from the center of the sphere. For convenience, we let the line from P to O be the polar axis of a spherical coordinate system r, θ, ψ. The gravitational acceleration at P due to an element of mass dm located in the sphere at r', θ', ψ' is directed along the line from P to dm and is given by Equation (5–3). The component of this gravitational acceleration along the line from P to O is

$$\frac{G \cos \alpha \, dm}{b^2}.$$

The net radially inward gravitational acceleration at P is found by integrating this expression over the entire mass distribution:

$$g_m = G \int \frac{\cos \alpha \, dm}{b^2}. \tag{5.5}$$

The element of mass dm is the product of the volume element dV with the density $\rho(r')$ at the location of dV

$$dm = \rho(r') \, dV. \tag{5.6}$$

The element of volume can be expressed in spherical coordinates as

$$dV = r'^2 \sin\theta' \, d\theta' \, d\psi' \, dr'. \tag{5.7}$$

The integral over the spherical mass distribution in Equation (5–5) can thus be written

$$g_m = G \int_0^a \int_0^\pi \int_0^{2\pi} \frac{\rho(r') r'^2 \sin\theta' \cos\alpha \, d\psi' \, d\theta' \, dr'}{b^2}, \tag{5.8}$$

where a is the radius of the model Earth. The integral over ψ' is 2π, since the quantities in the integrand of Equation (5–8) are independent of ψ'. To carry out the integration over r' and θ', we need an expression for $\cos\alpha$. From the law of cosines we can write

$$\cos\alpha = \frac{b^2 + r^2 - r'^2}{2rb}. \tag{5.9}$$

Because the expression for cos α involves b rather than θ', it is more convenient to rewrite Equation (5–8) so that the integration can be carried out over b rather than over θ'. The law of cosines can be used again to find an expression for cos θ':

$$\cos\theta' = \frac{r'^2 + r^2 - b^2}{2rr'}. \tag{5.10}$$

By differentiating Equation (5–10) with r and r' held constant, we find

$$\sin\theta' d\theta' = \frac{b \, db}{rr'}. \tag{5.11}$$

Upon substitution of Equations (5–9) and (5–11) into Equation (5–8), we can write the integral expression for g_m as

$$g_m = \frac{\pi G}{r^2} \int_0^a r' \rho(r') \int_{r-r'}^{r+r'} \left\{ \frac{r^2 - r'^2}{b^2} + 1 \right\} db \, dr'. \tag{5.12}$$

The integration over b gives 4 r' so that Equation (5–12) becomes

$$g_m = \frac{4\pi G}{r^2} \int_0^a dr' r'^2 \rho(r'). \tag{5.13}$$

Since the total mass of the model is given by

$$M = 4\pi \int_0^a dr' r'^2 \rho(r'), \tag{5.14}$$

the gravitational acceleration is

$$g_m = \frac{GM}{r^2}. \tag{5.15}$$

The gravitational acceleration of a spherically symmetric mass distribution, at a point outside the mass, is identical to the acceleration obtained by concentrating all the mass at the center of the distribution. Even though there are lateral density variations in the Earth and the Earth's shape is distorted by rotation, the direction of the gravitational acceleration at a point external to the Earth is very nearly radially inward toward the Earth's center of mass, and Equation (5–15) provides an excellent first approximation for g_m.

Problem 5.1 For a point on the surface of the Moon determine the ratio of the acceleration of gravity due to the mass of the Earth to the acceleration of gravity due to the mass of the Moon.

The rotational distortion of the Earth's mass adds a small latitude-dependent term to the gravitational acceleration. This term depends on the excess mass in the rotational equatorial bulge of the Earth. The observed latitude dependence of g_m can thus be used to determine this excess mass. In addition, this effect must be removed from observed variations in surface gravity before the residual gravity anomalies can properly be attributed to density anomalies in the Earth's interior. The model we use to calculate the contribution of rotational distortion to gravitational acceleration is sketched in Figure 5–3. The Earth is assumed to be flattened at the poles and bulged at the equator because of its rotation with angular velocity ω. The mass distribution is assumed to be symmetrical about the rotation axis. Because of the departure from spherical symmetry due to rotation, the gravitational acceleration at a point P outside the Earth has both radial and tangential components. The radial component is the sum of GM/r^2 and the term g_r' due to rotational distortion of the mass distribution; the tangential component g_t' is entirely due to the rotationally induced departure from spherical symmetry. Following our previous sign convention both GM/r^2 and g_r' are positive if directed inward. Since rotation modifies the otherwise spherically symmetric model Earth only slightly, g_r' and g_t' are small compared with GM/r^2.
The total gravitational acceleration is

$$\begin{aligned} &\left\{\left(\frac{GM}{r^2} + g_r'\right)^2 + g_t'^2\right\}^{1/2} \\ &\quad = \left\{\left(\frac{GM}{r^2}\right)^2 + 2\left(\frac{GM}{r^2}\right)g_r' + g_r'^2 + g_t'^2\right\}^{1/2}. \end{aligned} \tag{5.16}$$

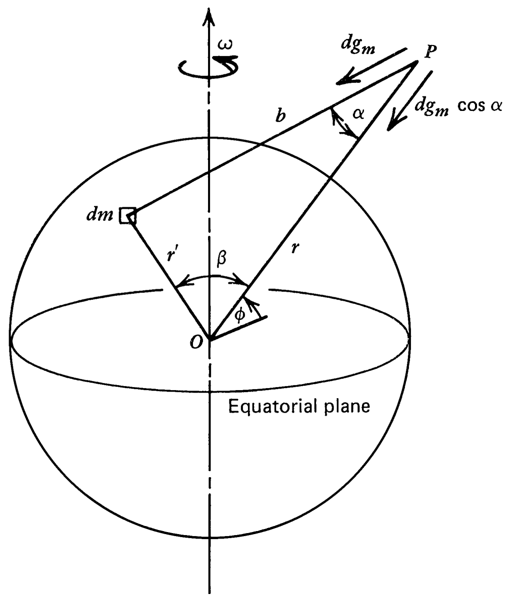

Figure 5.3 Geometry for calculating the contribution of rotational distortion to the gravitational acceleration.

It is appropriate to neglect the quadratic terms because the magnitudes of g_r' and g_t' are much less than GM/r^2. Therefore the gravitational acceleration is given by

$$\begin{aligned} &\left\{\left(\frac{GM}{r^2}\right)^2 + 2\left(\frac{GM}{r^2}\right)g_r'\right\}^{1/2} \\ &= \left(\frac{GM}{r^2}\right)\left\{1 + \frac{2g_r'}{GM/r^2}\right\}^{1/2} \\ &= \left(\frac{GM}{r^2}\right)\left\{1 + \frac{g_r'}{GM/r^2}\right\} = \frac{GM}{r^2} + g_r'. \end{aligned} \tag{5.17}$$

Equation (5–17) shows that the tangential component of the gravitational acceleration is negligible; the net gravitational acceleration at a point P external to a rotationally distorted model Earth is essentially radially inward to the center of the mass distribution.

The radial gravitational acceleration for the rotationally distorted Earth model can be obtained by integrating Equation (5–5) over the entire mass distribution. We can rewrite this equation for g_m by substituting expression (5–9) for cos α with the result

$$g_m = \frac{G}{2r^2}\int\left\{\frac{r}{b} + \frac{r^3}{b^3}\left(1 - \frac{r'^2}{r^2}\right)\right\}dm. \tag{5.18}$$

The three distances appearing in the integral of Equation (5–18) r, r', and b are the sides of the triangle connecting O, P, and dm in Figure 5–3. It is helpful for carrying out the integration to eliminate b from the integrand in terms of r, r', and the angle β, which is opposite the side of length b in this triangle. From the law of cosines we can write

$$b^2 = r^2 + r'^2 - 2rr'\cos\beta, \tag{5.19}$$

which can be rearranged as

$$\frac{r}{b} = \left\{1 + \frac{r'^2}{r^2} - \frac{2r'}{r}\cos\beta\right\}^{-1/2}. \tag{5.20}$$

Upon substituting Equation (5–20) into Equation (5–18), we get

$$g_m = \frac{G}{2r^2}\int\left\{1 + \frac{r'^2}{r^2} - \frac{2r'}{r}\cos\beta\right\}^{-1/2} \times\left\{1 + \left(1 - \frac{r'^2}{r^2}\right)\left(1 + \frac{r'^2}{r^2} - \frac{2r'}{r}\cos\beta\right)^{-1}\right\} dm. \tag{5.21}$$

An analytic evaluation of the integral in Equation (5–21) is not possible. The integration is complicated because both r' and β vary with the position of dm. However, the integration can be made tractable by approximating the integrand with a power series in r'/r and retaining terms only up to $(r'/r)^2$. For P outside the mass distribution, $r'/r < 1$. We will show that the expansion in powers of (r'/r) is equivalent to an expansion of the gravitational field in powers of a/r. This approximation yields an expression for g_m that is sufficiently accurate for our purposes. Using the formulas

$$(1+\varepsilon)^{-1/2} \approx 1 - \frac{\varepsilon}{2} + \frac{3\varepsilon^2}{8} + \cdots \tag{5.22}$$

$$(1+\varepsilon)^{-1} \approx 1 - \varepsilon + \varepsilon^2 + \cdots, \tag{5.23}$$

which are approximately valid for $\varepsilon < 1$, we find

$$g_m = \frac{G}{r^2}\int\left\{1 + \frac{2r'}{r}\cos\beta + \frac{3r'^2}{r^2}\left(1 - \frac{3}{2}\sin^2\beta\right)\right\} dm. \tag{5.24}$$

The integrations in Equation (5–24) can be carried out in terms of well-known physical properties of a mass distribution. The first term is just the integral of dm over the entire mass. The result is simply M. The integral of $r'\cos\beta$ over the mass distribution is a first moment of the distribution.

It is by definition zero if the origin of the coordinate system is the center of mass of the distribution. Thus Equation (5–24) becomes

$$g_m = \frac{GM}{r^2} + \frac{3G}{r^4} \int r'^2 \left(1 - \frac{3}{2} \sin^2 \beta\right) dm. \tag{5.25}$$

The first term on the right of Equation (5–25) is the gravitational acceleration of a spherically symmetric mass distribution. The second term is the modification due to rotationally induced *oblateness* of the body. If higher order terms in Equations (5–24) and (5–23) had been retained, the expansion given in Equation (5–25) would have been extended to include terms proportional to r^{-5} and higher powers of r^{-1}.

We will now express the integral appearing in Equation (5–25) in terms of the moments of inertia of an axisymmetric body. We take C to be the moment of inertia of the body about the rotational or z axis defined by $\theta = 0$. This moment of inertia is the integral over the entire mass distribution of dm times the square of the perpendicular distance from dm to the rotational axis. The square of this distance is $x'^2 + y'^2$ so that we can write C as

$$C \equiv \int (x'^2 + y'^2)\, dm = \int r'^2 \sin^2 \theta' \, dm \tag{5.26}$$

because

$$x' = r' \sin \theta' \cos \psi' \tag{5.27}$$

$$y' = r' \sin \theta' \sin \psi'. \tag{5.28}$$

The moment of inertia about the x axis, which is defined by $\theta = \pi/2$, $\psi = 0$, is

$$\begin{aligned} A &\equiv \int (y'^2 + z'^2)\, dm \\ &= \int r'^2 (\sin^2 \theta' \sin^2 \psi' + \cos^2 \theta')\, dm \end{aligned} \tag{5.29}$$

because

$$z' = r' \cos \theta'. \tag{5.30}$$

Similarly, the moment of inertia about the y axis, which is defined by $\theta = \pi/2,\ \psi = \pi/2$, is

$$\begin{aligned} B &\equiv \int (x'^2 + z'^2)\, dm \\ &= \int r'^2 (\sin^2 \theta' \cos^2 \psi' + \cos^2 \theta')\, dm. \end{aligned} \tag{5.31}$$

For a body that is axisymmetric about the rotation or z axis, $A = B$. The addition of Equations (5–26), (5–29), and (5–31) together with the assumption of axisymmetry gives

$$A + B + C = 2\int r'^2\, dm = 2A + C. \tag{5.32}$$

This equation expresses the integral of $r'^2 dm$ appearing in Equation (5–25) in terms of the moments of inertia of the body.

We will next derive an expression for the integral of $r'^2 \sin^2 \beta dm$. Because of the axial symmetry of the body there is no loss of generality in letting the line OP in Figure 5–3 lie in the xz plane. With the help of Equation (5–32) we rewrite the required integral as

$$\begin{aligned}\int r'^2 \sin^2 \beta\, dm &= \int r'^2(1 - \cos^2 \beta)\, dm \\ &= A + \frac{1}{2}C - \int r'^2 \cos^2 \beta\, dm.\end{aligned} \tag{5.33}$$

The quantity $r' \cos \beta$ is the projection of r' along OP. But this is also

$$r' \cos \beta = x' \cos \phi + z' \sin \phi, \tag{5.34}$$

where ϕ is the latitude or the angle between OP and the xy plane. Note that y' has no projection onto OP, since OP is in the xz plane. We use Equation (5–34) to rewrite the integral of $r'^2 \cos^2 \beta$ in the form

$$\begin{aligned}\int r'^2 \cos^2 \beta\, dm = \cos^2 \phi &\int x'^2\, dm \\ + \sin^2 \phi &\int z'^2\, dm \\ +2 \cos \phi \sin \phi &\int x'z'\, dm.\end{aligned} \tag{5.35}$$

For an axisymmetric body,

$$\int x'^2\, dm = \int y'^2\, dm. \tag{5.36}$$

This result and Equation (5–26) give

$$\int x'^2\, dm = \frac{1}{2}\int (x'^2 + y'^2)\, dm = \frac{1}{2}C. \tag{5.37}$$

The integral of $z'^2 dm$ can be evaluated by using Equations (5–26) and (5–32)

$$\begin{aligned}\int z'^2\,dm &= \int (x'^2 + y'^2 + z'^2)\,dm - \int (x'^2 + y'^2)\,dm \\ &= \int r'^2\,dm - \int (x'^2 + y'^2)\,dm \\ &= A - \frac{1}{2}C. \end{aligned} \tag{5.38}$$

With mass symmetry about the equatorial plane we have

$$\int x'z'\,dm = \int r'^2 \cos\theta' \sin\theta' \cos\psi'\,dm = 0. \tag{5.39}$$

Substitution of Equations (5–37) to (5–39) into Equation (5–35) yields

$$\int r'^2 \cos^2\beta\,dm = \frac{1}{2}C\cos^2\phi + \left(A - \frac{1}{2}C\right)\sin^2\phi. \tag{5.40}$$

When Equations (5–33) and (5–40) are combined, we find, using $\sin^2\phi + \cos^2\phi = 1$, that

$$\int r'^2 \sin^2\beta\,dm = A\cos^2\phi + C\sin^2\phi. \tag{5.41}$$

The gravitational acceleration is finally obtained by substituting Equations (5–32) and (5–41) into Equation (5–25):

$$g_m = \frac{GM}{r^2} - \frac{3G(C-A)}{2r^4}(3\sin^2\phi - 1). \tag{5.42}$$

Equation (5–42) is a simplified form of *MacCullagh's formula* for an axisymmetric body. The moment of inertia about the rotational axis C is larger than the moment of inertia about an equatorial axis A because of the rotational flattening of the body. It is customary to write the difference in moments of inertia as a fraction J_2 of Ma^2, that is

$$C - A = J_2 M a^2, \tag{5.43}$$

where a is the Earth's equatorial radius. In terms of J_2, g_m is

$$g_m = \frac{GM}{r^2} - \frac{3GMa^2 J_2}{2r^4}(3\sin^2\phi - 1). \tag{5.44}$$

The Earth's gravitational field can be accurately determined from the tracking of artificial satellites. The currently accepted values are:

$$\begin{aligned} a &= 6378.137 \text{ km} \\ GM &= 3.98600440 \times 10^{14} \text{ m}^3\text{s}^{-2} \end{aligned}$$

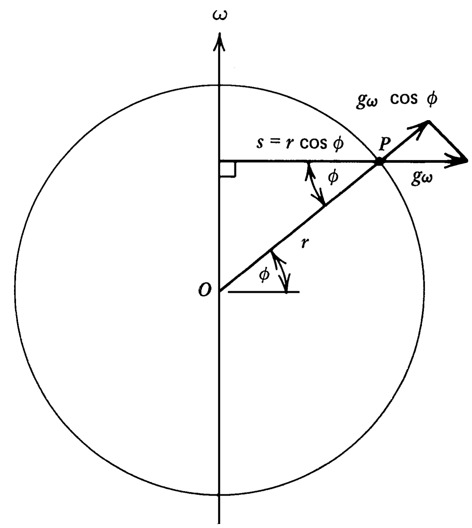

Figure 5.4 Centrifugal acceleration at a point on the Earth's surface.

$$J_2 = 1.0826265 \times 10^{-3}. \tag{5.45}$$

Although a satellite is acted upon only by the Earth's gravitational acceleration, an object on the Earth's surface is also subjected to a centrifugal acceleration due to the Earth's rotation.

5.3 Centrifugal Acceleration and the Acceleration of Gravity

The force on a unit mass at the surface of the Earth due to the rotation of the Earth with angular velocity ω is the *centrifugal acceleration* g_ω. It points radially outward along a line perpendicular to the rotation axis and passing through P, as shown in Figure 5–4, and is given by

$$g_\omega = \omega^2 s, \tag{5.46}$$

where s is the perpendicular distance from P to the rotation axis. If r is the radial distance from P to the center of the Earth and ϕ is the latitude of point P, then

$$s = r\cos\phi \tag{5.47}$$

and

$$g_\omega = \omega^2 r \cos\phi. \tag{5.48}$$

The currently accepted value for the Earth's angular velocity is

$$\omega = 7.292115 \times 10^{-5} \text{ rad s}^{-1}.$$

Problem 5.2 Determine the ratio of the centrifugal acceleration to the gravitational acceleration at the Earth's equator.

The gravitational and centrifugal accelerations of a mass at the Earth's surface combine to yield the acceleration of gravity g. Because $g_\omega \ll g_m$, it is appropriate to add the radial component of the centrifugal acceleration to g_m to obtain g; see Equations (5–16) and (5–17). As shown in Figure 5–4, the radial component of centrifugal acceleration points radially outward. In agreement with our sign convention that inward radial accelerations are positive, the radial component of the centrifugal acceleration is

$$g_r' = -g_\omega \cos\phi = -\omega^2 r \cos^2\phi. \tag{5.49}$$

Therefore, the acceleration of gravity g is the sum of g_m in Equation (5–44) and g_r':

$$g = \frac{GM}{r^2} - \frac{3GMa^2J_2}{2r^4}(3\sin^2\phi - 1) - \omega^2 r\cos^2\phi. \tag{5.50}$$

Equation (5–50) gives the radially inward acceleration of gravity for a point located on the surface of the model Earth at latitude ϕ and distance r from the center of mass.

5.4 The Gravitational Potential and the Geoid

By virtue of its position in a gravitational field, a mass m' has *gravitational potential energy*. The energy can be regarded as the negative of the work done on m' by the gravitational force of attraction in bringing m' from infinity to its position in the field. The *gravitational potential* V is the potential energy of m' divided by its mass. Because the gravitational field is *conservative*, the potential energy per unit mass V depends only on the position in the field and not on the path through which a mass is brought to the location. To calculate V for the rotationally distorted model Earth, we can imagine bringing a unit mass from infinity to a distance r from the center of the model along a radial path. The negative of the work done on the unit mass by the gravitational field of the model is the integral of the product of the force per unit mass g_m in Equation (5–44) with the increment of distance dr (the acceleration of gravity and the increment dr are oppositely directed):

$$V = \int_\infty^r \left\{ \frac{GM}{r'^2} - \frac{3GMa^2J_2}{2r'^4}(3\sin^2\phi - 1) \right\} dr'$$

(5.51)

or

$$V = -\frac{GM}{r} + \frac{GMa^2J_2}{2r^3}(3\sin^2\phi - 1). \tag{5.52}$$

In evaluating V, we assume that the potential energy at an infinite distance from the Earth is zero. The gravitational potential adjacent to the Earth is negative; Earth acts as a potential well. The first term in Equation (5–52) is the gravitational potential of a point mass. It is also the gravitational potential outside any spherically symmetric mass distribution. The second term is the effect on the potential of the Earth model's rotationally induced oblateness. A *gravitational equipotential surface* is a surface on which V is a constant. Gravitational equipotentials are spheres for spherically symmetric mass distributions.

Problem 5.3 (*a*) What is the gravitational potential energy of a 1-kg mass at the Earth's equator? (*b*) If this mass fell toward the Earth from a large distance where it had zero relative velocity, what would be the velocity at the Earth's surface? (*c*) If the available potential energy was converted into heat that uniformly heated the mass, what would be the temperature of the mass if its initial temperature $T_0 = 100$ K, $c = 1$ kJ kg^{-1} K^{-1}, $T_m = 1500$ K, and $L = 400$ kJ kg^{-1}?

A comparison of Equations (5–44) and (5–52) shows that V is the integral of the radial component of the gravitational acceleration g_m with respect to r. To obtain a *gravity potential U* which accounts for both gravitation and the rotation of the model Earth, we can take the integral with respect to r of the radial component of the acceleration of gravity g in Equation (5–50) with the result that

$$\begin{aligned} U = &-\frac{GM}{r} + \frac{GMa^2J_2}{2r^3}(3\sin^2\phi - 1) \\ &- \frac{1}{2}\omega^2 r^2\cos^2\phi. \end{aligned} \tag{5.53}$$

A *gravity equipotential* is a surface on which U is a constant. Within a few meters the sea surface defines an equipotential surface. Therefore, elevations above or below sea level are distances above or below a reference equipotential surface.

The reference equipotential surface that defines sea level is called the *geoid*. We will now obtain an expression for the geoid surface that is consistent with our second-order expansion of the gravity potential given in Equation (5–53). The value of the surface gravity potential at the equator is found by substituting $r = a$ and $\phi = 0$ in Equation (5–53) with the result

$$U_0 = -\frac{GM}{a}\left(1 + \frac{1}{2}J_2\right) - \frac{1}{2}a^2\omega^2. \tag{5.54}$$

The value of the surface gravity potential at the poles must also be U_0 because we define the surface of the model Earth to be an equipotential surface. We substitute $r = c$ (the Earth's polar radius) and $\phi = \pm\pi/2$ into Equation (5–53) and obtain

$$U_0 = -\frac{GM}{c}\left[1 - J_2\left(\frac{a}{c}\right)^2\right]. \tag{5.55}$$

The *flattening* (*ellipticity*) of this geoid is defined by

$$f \equiv \frac{a - c}{a}. \tag{5.56}$$

The flattening is very slight; that is, $f \ll 1$. In order to relate the flattening f to J_2, we set Equations (5–54) and (5–55) equal and obtain

$$1 + \frac{1}{2}J_2 + \frac{1}{2}\frac{a^3\omega^2}{GM} = \frac{a}{c}\left[1 - J_2\left(\frac{a}{c}\right)^2\right]. \tag{5.57}$$

Substituting $c = a(1 - f)$ and the neglecting quadratic and higher order terms in f and J_2, because $f \ll 1$ and $J_2 \ll 1$, we find that

$$f = \frac{3}{2}J_2 + \frac{1}{2}\frac{a^3\omega^2}{GM}. \tag{5.58}$$

Taking $a^3\omega^2/GM = 3.46139 \times 10^{-3}$ and $J_2 = 1.0826265 \times 10^{-3}$ from Equation (5–45), we find from Equation (5–58) that $f = 3.3546 \times 10^{-3}$. Retention of higher order terms in the theory gives the more accurate value

$$f = 3.35281068 \times 10^{-3} = \frac{1}{298.257222}. \tag{5.59}$$

It should be emphasized that Equation (5–58) is valid only if the surface of the planetary body is an equipotential.

The shape of the model geoid is nearly that of a spherical surface; that is, if r_0 is the distance to the geoid,

$$r_0 \approx a(1 - \varepsilon), \tag{5.60}$$

where $\varepsilon \ll 1$. By setting $U = U_0$ and $r = r_0$ in Equation (5–53), substituting Equation (5–54) for U_0 and Equation (5–60) for r_0, and neglecting quadratic and higher order terms in f, J_2, $a^3\omega^2/GM$, and ε, we obtain

$$\varepsilon = \left(\frac{3}{2}J_2 + \frac{1}{2}\frac{a^3\omega^2}{GM}\right)\sin^2\phi. \tag{5.61}$$

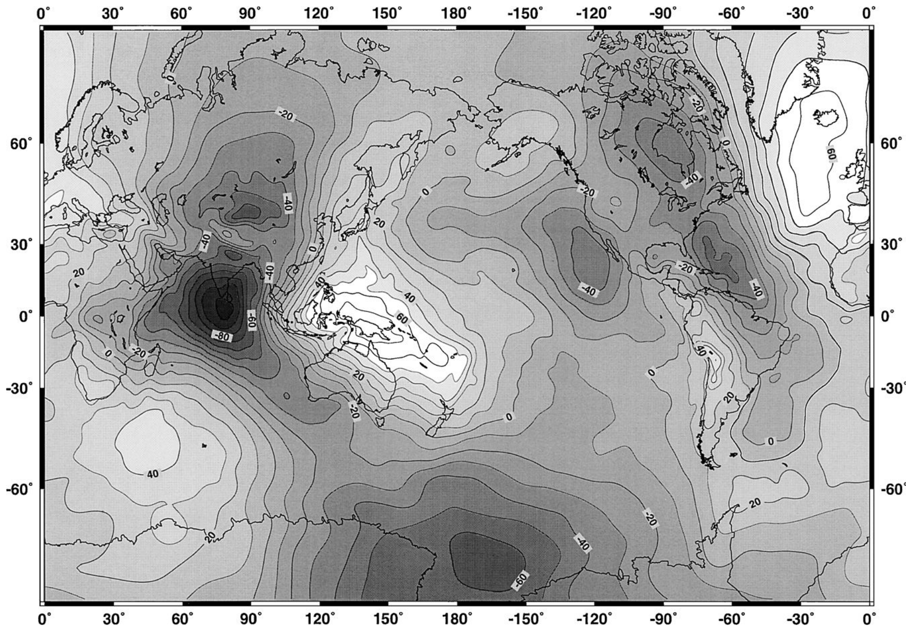

Figure 5.5 Geoid height (EGM96) above reference ellipsoid WGS84 (Lemoine et al., 1998).

The substitution of Equation (5–61) into Equation (5–60) gives the approximate model equation for the geoid as

$$r_0 = a\left\{1 - \left(\frac{3}{2}J_2 + \frac{1}{2}\frac{a^3\omega^2}{GM}\right)\sin^2\phi\right\} \tag{5.62}$$

or

$$r_0 = a(1 - f\sin^2\phi). \tag{5.63}$$

The nondimensional quantity $a^3\omega^2/GM$ is a measure of the relative importance of the centrifugal acceleration due to the rotation of the Earth compared with the gravitational attraction of the mass in the Earth. The rotational contribution is about 0.33% of the mass contribution.

In the preceding analysis we considered only terms linear in J_2 and $a^3\omega^2/GM$. In order to provide a reference geoid against which geoid anomalies are measured, it is necessary to include higher order terms. By convention, the *reference geoid* is a *spheroid* (*ellipsoid of revolution*) defined in terms of the equatorial and polar radii by

$$\frac{r_0^2\cos^2\phi}{a^2} + \frac{r_0^2\sin^2\phi}{c^2} = 1. \tag{5.64}$$

The *eccentricity* e of the spheroid is given by

$$e \equiv \left(\frac{a^2 - c^2}{a^2}\right)^{1/2} = (2f - f^2)^{1/2}. \tag{5.65}$$

It is the usual practice to express the reference geoid in terms of the equatorial radius and the flattening with the result

$$\frac{r_0^2 \cos^2\phi}{a^2} + \frac{r_0^2 \sin^2\phi}{a^2(1-f)^2} = 1 \tag{5.66}$$

or

$$r_0 = a\left[1 + \frac{(2f - f^2)}{(1-f)^2}\sin^2\phi\right]^{-1/2}. \tag{5.67}$$

If Equation (5–67) is expanded in powers of f and if terms of quadratic and higher order in f are neglected, the result agrees with Equation (5–63). Equation (5–67) with $a = 6378.137$ km and $f = 1/298.257222$ defines the reference geoid.

The difference in elevation between the measured geoid and the reference geoid ΔN is referred to as a *geoid anomaly.* A map of geoid anomalies is given in Figure 5–5. The maximum geoid anomalies are around 100 m; this is about 0.5% of the 21-km difference between the equatorial and polar radii. Clearly, the measured geoid is very close to having the spheroidal shape of the reference geoid.

The major geoid anomalies shown in Figure 5–5 can be attributed to density inhomogeneities in the Earth. A comparison with the distribution of surface plates given in Figure 1–1 shows that some of the major anomalies can be directly associated with plate tectonic phenomena. Examples are the geoid highs over New Guinea and Chile–Peru; these are clearly associated with subduction. The excess mass of the dense subducted lithosphere causes an elevation of the geoid. The negative geoid anomaly over China may be associated with the continental collision between the Indian and Eurasian plates and the geoid low over the Hudson Bay in Canada may be associated with postglacial rebound (see Section 6–10). The largest geoid anomaly is the negative geoid anomaly off the southern tip of India, which has an amplitude of 100 m. No satisfactory explanation has been given for this geoid anomaly, which has no surface expression. A similar unexplained negative geoid anomaly lies off the west coast of North America.

The definition of geoid anomalies relative to the reference geoid is somewhat arbitrary. The reference geoid itself includes an averaging over density anomalies within the Earth. An alternative approach is to define geoid anomalies relative to a hydrostatic geoid. The Earth is assumed to have a

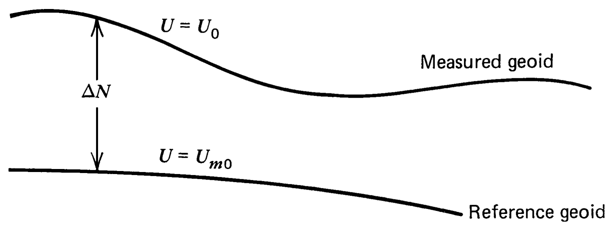

Figure 5.6 Relationship of measured and reference geoids and geoid anomaly Δ N.

layered structure in terms of density, but each layer is in hydrostatic equilibrium relative to the rotation of the Earth. The anomaly map is significantly different for the two approaches, but the major features remain unaffected.

One of the primary concerns in geodesy is to define topography and bathymetry. Both are measured relative to "sea level." Sea level is closely approximated by an equipotential surface corresponding to a constant value of U. As we have discussed, geoid anomalies relative to a reference spheroidal surface can be as large as 100 m. Thus, if we define sea level by a global spheroid we would be in error by this amount. Topography (and bathymetry) in any local area must be measured relative to a surface that approximates the local sea level (equipotential surface).

Problem 5.4 Assume a large geoid anomaly with a horizontal scale of several thousand kilometers has a mantle origin and its location does not change. Because of continental drift the passive margin of a continent passes through the anomaly. Is there a significant change in sea level associated with the passage of the margin through the geoid anomaly? Explain your answer.

The anomaly in the potential of the gravity field measured on the reference geoid ΔU can be related directly to the geoid anomaly ΔN. The *potential anomaly* is defined by

$$\Delta U = U_{m0} - U_0, \tag{5.68}$$

where U_{m0} is the measured potential at the location of the reference geoid and U_0 is the reference value of the potential defined by Equation (5–54). The potential on the measured geoid is U_0, as shown in Figure 5–6. It can be seen from the figure that U_0, U_{m0}, and ΔN are related by

$$U_0 = U_{m0} + \left(\frac{\partial U}{\partial r}\right)_{r=r_0} \Delta N, \tag{5.69}$$

because $\Delta N/a \ll 1$. Recall from the derivation of Equation (5–53) that we obtained the potential by integrating the acceleration of gravity. Therefore, the radial derivative of the potential in Equation (5–69) is the acceleration of gravity on the reference geoid. To the required accuracy we can write

$$\left(\frac{\partial U}{\partial r}\right)_{r=r_0} = g_0, \tag{5.70}$$

where g_0 is the reference acceleration of gravity on the reference geoid. Just as the measured potential on the reference geoid differs from U_0, the measured acceleration of gravity on the reference geoid differs from g_0. However, for our purposes we can use g_0 in Equation (5–69) for $(\partial U/\partial r)_{r=r_0}$ because this term is multiplied by a small quantity ΔN. Substitution of Equations (5–69) and (5–70) into Equation (5–68) gives

$$\Delta U = -g_0 \Delta N. \tag{5.71}$$

A local mass excess produces an outward warp of gravity equipotentials and therefore a positive ΔN and a negative ΔU. Note that the measured geoid essentially defines sea level. Deviations of sea level from the equipotential surface are due to lunar and solar tides, winds, and ocean currents. These effects are generally a few meters.

The reference acceleration of gravity on the reference geoid is found by substituting the expression for r_0 given by Equation (5–62) into Equation (5–50) and simplifying the result by neglecting quadratic and higher order terms in J_2 and $a^3\omega^2/GM$. One finds

$$g_0 = \frac{GM}{a^2}\left(1 + \frac{3}{2}J_2 \cos^2\phi\right) + a\omega^2(\sin^2\phi - \cos^2\phi). \tag{5.72}$$

To provide a standard *reference acceleration of gravity* against which gravity anomalies are measured, we must retain higher order terms in the equation for g_0. Gravity anomalies are the differences between measured values of g on the reference geoid and g_0. By international agreement in 1980 the reference gravity field was defined to be

$$\begin{aligned} g_0 = \ & 9.7803267715(1 + 0.0052790414 \sin^2\phi \\ & + 0.0000232718 \sin^4\phi \\ & + 0.0000001262 \sin^6\phi \\ & + 0.0000000007 \sin^8\phi), \end{aligned} \tag{5.73}$$

with g_0 in m s^{-2}. This is known as the 1980 *Geodetic Reference System* (*GRS*) (*80*) *Formula*. The standard reference gravity field given by Equation (5–73) is of higher order in ϕ than is the consistent quadratic approximation used to specify both g_0 in Equation (5–72) and r_0 in Equation (5–67). The suitable SI unit for gravity anomalies is mm s^{-2}.

Problem 5.5 Determine the values of the acceleration of gravity at the equator and the poles using GRS 80 and the quadratic approximation given in Equation (5–72).

Problem 5.6 By neglecting quadratic and higher order terms, show that the gravity field on the reference geoid can be expressed in terms of the gravity field at the equator g_e according to

$$g_0 = g_e \left[1 + \left(2\frac{\omega^2 a^3}{GM} - \frac{3}{2} J_2 \right) \sin^2 \phi \right]. \tag{5.74}$$

Problem 5.7 What is the value of the acceleration of gravity at a distance b above the geoid at the equator ($b \ll a$)?

5.5 Moments of Inertia

MacCullagh's formula given in Equation (5–42) relates the gravitational acceleration of an oblate planetary body to its principal moments of inertia. Thus, we can use the formula, together with measurements of a planet's gravitational field by flyby or orbiting spacecraft, for example, to constrain the moments of inertia of a planet. Since the moments of inertia reflect a planet's overall shape and internal density distribution, we can use the values of the moments to learn about a planet's internal structure. For this purpose it is helpful to have expressions for the moments of inertia of some simple bodies such as spheres and spheroids.

The principal moments of inertia of a spherically symmetric body are all equal, $A = B = C$, because the mass distribution is the same about any axis passing through the center of the body. For simplicity, we will determine the moment of inertia about the polar axis defined by $\theta = 0$. For a spherical body of radius a, substitution of Equations (5–6) and (5–7) into Equation (5–26) gives

$$C = \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho(r') r'^4 \sin^3 \theta' \, dr' \, d\theta' \, d\psi'. \tag{5.75}$$

Integration over the angles ψ' and θ' results in

$$\int_0^{2\pi} d\psi' = 2\pi$$

and

$$\int_0^{\pi} \sin^3 \theta' \, d\theta' = \left[\frac{1}{3} \cos^3 \theta' - \cos \theta' \right]_0^{\pi} = \frac{4}{3},$$

Table 5.1 *Values of the Dimensionless Polar Moment of Inertia, J_2, and the Polar Flattening for the Earth, Moon, Mars, and Venus*

	Earth	Moon	Mars	Venus
C/Ma^2	0.3307007	0.3935	0.366	0.33
$J_2 \equiv \frac{1}{Ma^2}\left(C - \frac{A+B}{2}\right)$	1.0826265×10^{-3}	2.037×10^{-4}	1.96045×10^{-3}	4.458×10^{-6}
$f \equiv \frac{2}{(a+b)}\left(\frac{a+b}{2} - c\right)$	$3.35281068 \times 10^{-3}$	1.247×10^{-3}	6.4763×10^{-3}	——

so that Equation (5–75) becomes

$$C = \frac{8\pi}{3}\int_0^a \rho(r')r'^4\,dr'. \tag{5.76}$$

For a spherical body with a constant density ρ_0, the integration of Equation (5–76) gives

$$C = \frac{8\pi}{15}\rho_0 a^5. \tag{5.77}$$

Because the mass of the sphere is

$$M = \frac{4}{3}\pi a^3 \rho_0, \tag{5.78}$$

the moment of inertia is also given by

$$C = \frac{2}{5}Ma^2. \tag{5.79}$$

The dimensionless polar moments of inertia of the Earth and Moon are listed in Table 5–1. The value $C/Ma^2 = 0.3307$ for the Earth is considerably less than the value 0.4 that Equation (5–79) gives for a constant-density spherical planet. This difference is clearly associated with the Earth's high-density core. The value $C/Ma^2 = 0.3935$ for the Moon is close to the value for a constant-density planet, but does not rule out a small (radius less than about 300 km) metallic core.

Problem 5.8 Consider a spherical body of radius a with a core of radius r_c and constant density ρ_c surrounded by a mantle of constant density ρ_m. Show that the moment of inertia C and mass M are given by

$$C = \frac{8\pi}{15}[\rho_c r_c^5 + \rho_m(a^5 - r_c^5)] \tag{5.80}$$

$$M = \frac{4\pi}{3}[\rho_c r_c^3 + \rho_m(a^3 - r_c^3)]. \tag{5.81}$$

Determine mean values for the densities of the Earth's mantle and core given

$C = 8.04 \times 10^{37}$ kg m^2, $M = 5.97 \times 10^{24}$ kg, $a = 6378$ km, and $r_c = 3486$ km.

We will next determine the principal moments of inertia of a constant-density spheroid defined by

$$r_0 = \frac{ac}{(a^2 \cos^2 \theta + c^2 \sin^2 \theta)^{1/2}}. \tag{5.82}$$

This is a rearrangement of Equation (5–64) with the colatitude θ being used in place of the latitude ϕ. By substituting Equations (5–6) and (5–7) into Equations (5–26) and (5–29), we can write the polar and equatorial moments of inertia as

$$C = \rho \int_0^{2\pi} \int_0^{r_0} \int_0^{\pi} r'^4 \sin^3 \theta' \, d\theta' \, dr' \, d\psi' \tag{5.83}$$

$$A = \rho \int_0^{2\pi} \int_0^{r_0} \int_0^{\pi} r'^4 \sin \theta' \times (\sin^2 \theta' \sin^2 \psi' + \cos^2 \theta') \, d\theta' \, dr' \, d\psi', \tag{5.84}$$

where the upper limit on the integral over r' is given by Equation (5–82) and $B = A$ for this axisymmetric body. The integrations over ψ' and r' are straightforward and yield

$$C = \frac{2}{5} \pi \rho a^5 c^5 \int_0^{\pi} \frac{\sin^3 \theta' \, d\theta'}{(a^2 \cos^2 \theta' + c^2 \sin^2 \theta')^{5/2}} \tag{5.85}$$

$$A = \frac{1}{2} C + \frac{2}{5} \pi \rho a^5 c^5 \int_0^{\pi} \frac{\cos^2 \theta' \sin \theta' \, d\theta'}{(a^2 \cos^2 \theta' + c^2 \sin^2 \theta')^{5/2}}. \tag{5.86}$$

The integrals over θ' can be simplified by introducing the variable $x = \cos \theta' (dx = -\sin \theta' \, d\theta', \sin \theta' = (1 - x^2)^{1/2})$ with the result

$$C = \frac{2}{5} \pi \rho a^5 c^5 \int_{-1}^{1} \frac{(1 - x^2) \, dx}{[c^2 + (a^2 - c^2) x^2]^{5/2}} \tag{5.87}$$

$$A = \frac{1}{2} C + \frac{2}{5} \pi \rho a^5 c^5 \int_{-1}^{1} \frac{x^2 \, dx}{[c^2 + (a^2 - c^2) x^2]^{5/2}}. \tag{5.88}$$

From a comprehensive tabulation of integrals we find

$$\int_{-1}^{1} \frac{dx}{\{c^2 + (a^2 - c^2) x^2\}^{5/2}} = \frac{2}{3} \frac{(2a^2 + c^2)}{c^4 a^3} \tag{5.89}$$

$$\int_{-1}^{1} \frac{x^2\,dx}{\{c^2 + (a^2 - c^2)x^2\}^{5/2}} = \frac{2}{3}\frac{1}{c^2 a^3}. \tag{5.90}$$

By substituting Equations (5–89) and (5–90) into Equations (5–87) and (5–88), we obtain

$$C = \frac{8}{15}\pi\rho a^4 c \tag{5.91}$$

$$A = \frac{4}{15}\pi\rho a^2 c(a^2 + c^2). \tag{5.92}$$

These expressions for the moments of inertia can be used to determine J_2 for the spheroid. The substitution of Equations (5–91) and (5–92) into the definition of J_2 given in Equation (5–43), together with the equation for the mass of a constant-density spheroid

$$M = \frac{4\pi}{3}\rho a^2 c, \tag{5.93}$$

yields

$$J_2 = \frac{1}{5}\left(1 - \frac{c^2}{a^2}\right). \tag{5.94}$$

Consistent with our previous assumption that $J_2 \ll 1$ and $(1 - c/a) \ll 1$ this reduces to

$$J_2 = \frac{2}{5}\left(1 - \frac{c}{a}\right) = \frac{2f}{5}. \tag{5.95}$$

Equation (5–95) relates J_2 to the flattening of a constant-density planetary body. The deviation of the near-surface layer from a spherical shape produces the difference in polar and equatorial moments of inertia in such a body. For a planet that does not have a constant density, the deviation from spherical symmetry of the density distribution at depth also contributes to the difference in moments of inertia.

If the planetary surface is also an equipotential surface, Equation (5–58) is valid. Substitution of Equation (5–95) into that relation gives

$$f = \frac{5}{4}\frac{a^3\omega^2}{GM} \tag{5.96}$$

or

$$J_2 = \frac{1}{2}\frac{a^3\omega^2}{GM}. \tag{5.97}$$

These are the values of the flattening and J_2 expected for a constant-density, rotating planetary body whose surface is a gravity equipotential.

Observed values of J_2 and f are given in Table 5–1. For the Earth $J_2/f =$

0.3229 compared with the value 0.4 given by Equation (5–95) for a constant-density body. The difference can be attributed to the variation of density with depth in the Earth and the deviations of the density distribution at depth from spherical symmetry.

For the Moon, where a constant-density theory would be expected to be valid, $J_2/f = 0.16$. However, both J_2 and f are quite small. The observed difference in mean equatorial and polar radii is $(a + b)/2 - c = 2$ km, which is small compared with variations in lunar topography. Therefore the observed flattening may be influenced by variations in crustal thickness. Because the Moon is tidally coupled to the Earth so that the same side of the Moon always faces the Earth, the rotation of the Moon is too small to explain the observed value of J_2. However, the present flattening may be a relic of a time when the Moon was rotating more rapidly. At that time the lunar lithosphere may have thickened enough so that the strength of the elastic lithosphere was sufficient to preserve the rotational flattening.

For Mars, $a^3\omega^2/GM = 4.59 \times 10^{-3}$ and $J_2 = 1.960 \times 10^{-3}$. From Equation (5–58) the predicted value for the dynamic flattening is 5.235×10^{-3}. This compares with the observed flattening of 6.4763×10^{-3}. Again the difference may be attributed to the preservation of a fossil flattening associated with a higher rotational velocity in the past. The ratio of J_2 to the observed flattening is 0.3027; this again is considerably less than the value of 0.4 for a constant-density planet from Equation (5–95).

Problem 5.9 Assuming that the difference in moments of inertia $C - A$ is associated with a nearsurface density ρ_m and the mass M is associated with a mean planetary density $\bar{\rho}$, show that

$$J_2 = \frac{2}{5}\frac{\rho_m}{\bar{\rho}} f. \tag{5.98}$$

Determine the value of ρ_m for the Earth by using the measured values of J_2, $\bar{\rho}$, and f. Discuss the value obtained.

Problem 5.10 Assume that the constant-density theory for the moments of inertia of a planetary body is applicable to the Moon. Determine the rotational period of the Moon that gives the measured value of J_2.

Problem 5.11 Take the observed values of the flattening and J_2 for Mars and determine the corresponding period of rotation. How does this compare with the present period of rotation?

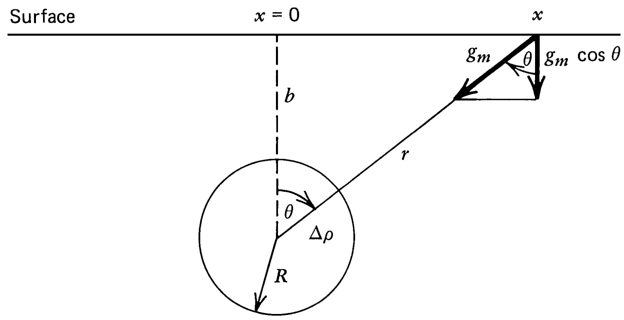

Figure 5.7 The gravitational attraction due to a sphere of anomalous density $\Delta\rho$ and radius R buried at a depth b beneath the surface.

5.6 Surface Gravity Anomalies

Mass anomalies on and in the Earth's crust are a primary source of surface gravity anomalies. Let us first consider the surface gravity anomalies caused by buried bodies of anomalous density. Examples include localized mineral deposits that usually have excess mass associated with them and igneous intrusions that often have an associated mass deficiency. The gravity anomaly due to a body of arbitrary shape and density distribution can be obtained by integrating Equation (5–3) over the body. However, it is generally impossible to carry out the necessary integrals except for the simplest shapes, and numerical methods are usually required.

As a specific example of a buried body we consider a buried sphere of radius R with a uniform density anomaly $\Delta\rho$, as shown in Figure 5–7. It should be emphasized that the effective density in determining the surface gravity anomaly caused by a buried body is the density difference between the body and the surrounding rock. From Equation (5–15), the gravitational acceleration due to the spherical mass anomaly at a distance r from its center ($r > R$) is

$$g_m = \frac{4\pi G R^3 \Delta\rho}{3r^2}. \tag{5.99}$$

This acceleration is directed toward the center of the sphere if $\Delta\rho$ is positive (see Figure 5–7). Because the gravitational acceleration due to the buried body is small compared with Earth's gravitational acceleration, the surface gravity anomaly Δg is just the vertical component of the surface gravitational acceleration of the body; see Equations (5–16) and (5–17). From Figure (5–7) we can write

$$\Delta g \equiv g_m \cos\theta, \tag{5.100}$$

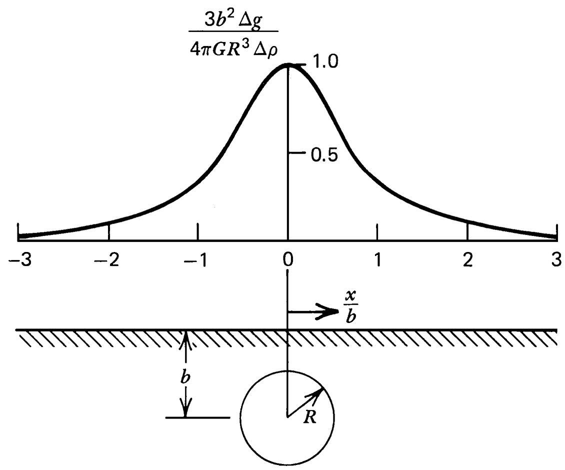

Figure 5.8 The surface gravity anomaly resulting from a spherical body of radius R whose center is at a depth b, as in Equation (5–102).

where θ is indicated in the figure. Gravity anomalies are measured positive downward. For a point on the surface,

$$\cos\theta = \frac{b}{r} = \frac{b}{(x^2+b^2)^{1/2}}, \tag{5.101}$$

where x is the horizontal distance between the surface point at which Δg is measured and the center of the sphere and b is the depth to the sphere's center. Substituting Equations (5–99) and (5–101) into Equation (5–100), we obtain

$$\Delta g = \frac{4\pi GR^3\Delta\rho b}{3r^3} = \frac{4\pi GR^3\Delta\rho}{3}\frac{b}{(x^2+b^2)^{3/2}}. \tag{5.102}$$

The resulting gravity anomaly is plotted in Figure 5–8.

A specific example of a surface gravity anomaly caused by a density anomaly at depth is the gravity anomaly over a salt dome off the Gulf coast of the United States. A contour map of the surface gravity anomaly is given in Figure 5–9*a*. Measurements of the gravity on the cross section *AA* are given in Figure 5–9*b*. The measurements are compared with the theoretical gravity anomaly computed from Equation (5–102) taking $b = 6$ km and $4\pi GR^3\Delta\rho/3b^2 = 0.1$ mm s^{-2}. Assuming that salt has a density of 2200 kg m^{-3} and that the mean density of the sediments is 2400 kg m^{-3}, we find that $R = 4.0$ km. This would appear to be a reasonable radius for an equivalent spherical salt dome.

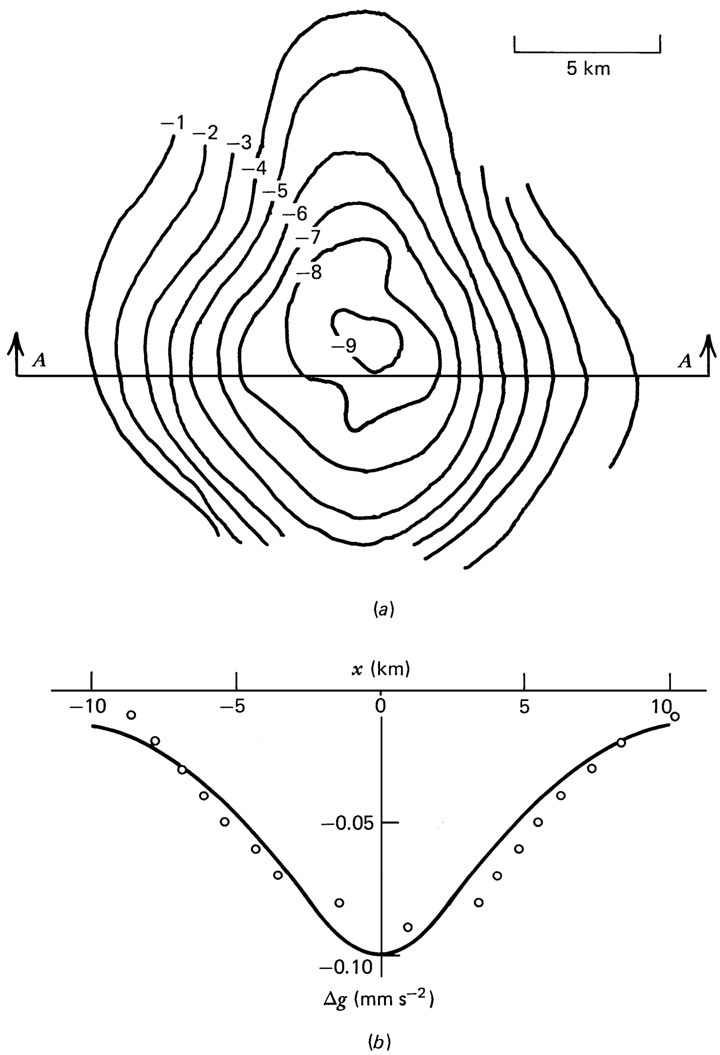

Figure 5.9 (a) Contour map (0.01 mm s^{-2} contours) of the surface gravity anomaly over a salt dome 125 miles southeast of Galveston, Texas, near the outer edge of the continental shelf (Nettleton, 1957). (b) Measurements of gravity on section AA from (a) compared with a theoretical fit based on Equation (5–102).

Problem 5.12 A gravity profile across the Pyramid No. 1 ore body near Pine Point, Northwest Territories, Canada, is shown in Figure 5–10. A reasonable fit with Equation (5–102) is obtained taking $b = 200$ m and $4\pi G R^3 \Delta\rho / 3b^2 = 0.006$ mm s^{-2}. Assume that the gravity anomaly is caused by lead–zinc ore with a density of 3650 kg m^{-3} and that the country rock has a density of 2650 kg m^{-3}. Estimate the tonnage of lead–zinc ore, assuming

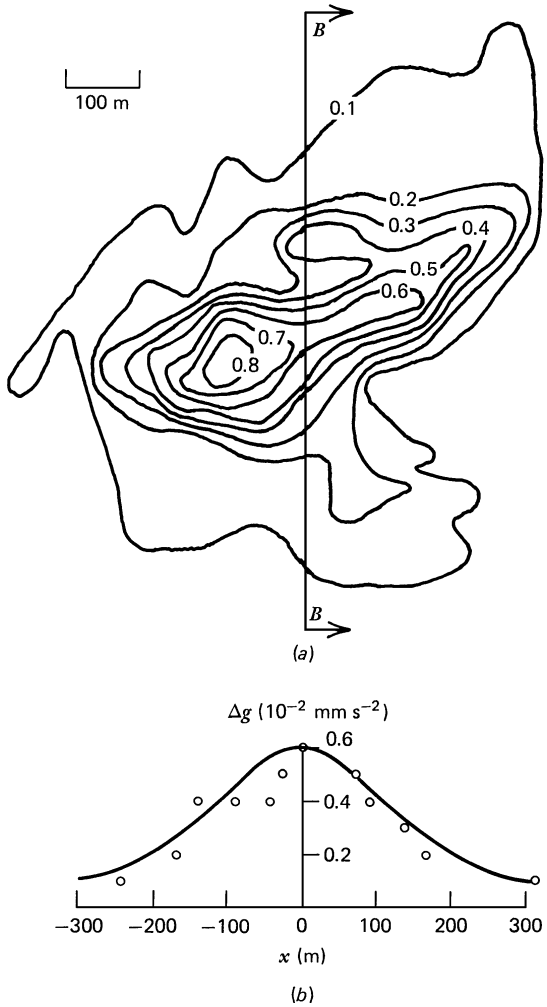

Figure 5.10 (a) Contour map (10^{-2} mm s^{-2} contours) of the surface gravity anomaly over the Pyramid No. 1 ore body (Seigel et al., 1968). (b) Gravity measurements on section BB from (a) compared with a theoretical fit based on Equation (5–102).

a spherical body. The tonnage established by drilling in this ore body was 9.2 million tons.

Problem 5.13 Show that the gravity anomaly of an infinitely long horizontal cylinder of radius R with anomalous density $\Delta\rho$ buried at depth b beneath the surface is

$$\Delta g = \frac{2\pi G R^2 \Delta\rho b}{(x^2 + b^2)}, \tag{5.103}$$

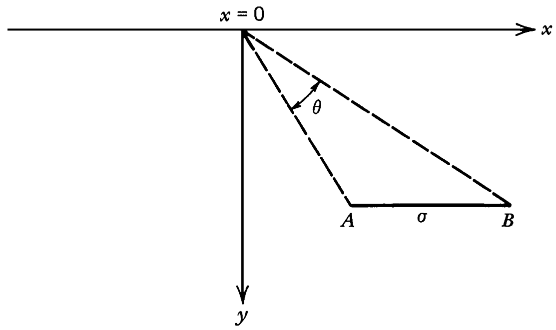

Figure 5.11 A buried sheet AB of excess mass σ per unit area.

where x is the horizontal distance from the surface measurement point to the point on the surface directly over the cylinder axis. What is the maximum gravity anomaly caused by a long horizontal underground tunnel of circular cross section with a 10-m radius driven through rock of density 2800 kg m^{-3} if the axis of the tunnel lies 50 m below the surface?

Problem 5.14 Calculate the gravity anomaly for a buried infinitely long horizontal line of excess mass γ per unit length by taking the limit of expression (5–103) as $R \to 0$ and $\Delta\rho \to \infty$ such that $\pi R^2 \Delta\rho \to \gamma$. The result is

$$\Delta g = \frac{2G\gamma b}{x^2 + b^2}, \tag{5.104}$$

where x is the horizontal distance from the surface observation point to the point directly above the line source. By integrating Equation (5–104), show that the gravity anomaly of a buried infinite slab of mass excess $\Delta\rho$ and thickness h is

$$\Delta g = 2\pi G h \Delta\rho. \tag{5.105}$$

Note that the anomaly of the infinite slab depends only on its density excess and thickness but not on its depth of burial.

Problem 5.15 Integrate Equation (5–104) to find the gravity anomaly, at $x = 0$, of the buried mass sheet shown in Figure 5–11. The sheet extends infinitely far in the z direction and has an excess density σ per unit area. The surface gravity anomaly at $x = 0$ is given by

$$\Delta g = 2G\sigma\theta, \tag{5.106}$$

where θ is the angle defined in Figure 5–11.